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Computational Contact Mechanics

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INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

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COMPUTATIONAL CONTACT MECHANICS

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PREFACE

In recent years, computational contact mechanics has been a topic of intense research. The aim of this research has been to devise robust solution schemes and new discretization techniques for description of contact phenomena, which can then be applied to a much broader range of engineering analysis areas than is currently the case. Among the broad areas of emphasis have been finite deformation contact problems, consideration of higher order interpolations, special algorithms for rolling contact, and discrete element problems for large scale analysis. The main focus of this book is to convey modern techniques applied within the range of computational contact mechanics.

Topics of interest within the community are wide ranging, and include computational aspects of

- *spatial and temporal discretization techniques for contact and impact problems with small and finite deformations;*
- *investigations on the reliability of micromechanical contact models;*
- *emerging techniques for rolling contact mechanics;*
- *homogenization methods and multi-scale approaches to frictionless and frictional contact problems;*
- *solution algorithms for single- and multi-processor computing environments, enabling methods that span from multi-contact to multi-scale approaches; and*
- *numerical experiments related to soil mechanics using discontinuous deformation analysis.*

The different contributions in this book will cover the topics described above, while providing some needed background with respect to continuum mechanics and finite element methods. The focus will be a detailed treatment of the theoretical formulation of contact problems with regard to mechanics and mathematics. Furthermore, discretization schemes for two- and three-dimensional contact problems of small and large deformations will be discussed. These schemes include novel and innovative formulations for rolling contact applications relating to tire contact and noise generation of rolling tires. Solution techniques related to contact mechanics are also of interest; hence, solvers for large scale multi-contact problems will be discussed. This includes multi-scale contact related to quasistatic, dynamic, structural and granular applications. Special attention is also given to conjugate gradient algorithms and extensions. This will include domain decomposition methods for structural problems, their application to cellular materials and important homogenization techniques in micro-macro approaches to frictional

problems. Finally, a contribution will cover discrete element techniques for multi body contact analysis and their applications to industrial problems.

All contributions are of a theoretical and applied nature, suitable for graduate students of applied mathematics, mechanics, engineering and physics with interest in computer simulation of contact problems.

P. Wriggers and T. A. Laursen

CONTENTS

Emerging Spatial and Temporal Discretization Methods in Contact and Impact Mechanics

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Abstract The focus of this discussion will be the recent evolution of both spatial and temporal discretization techniques in contact and impact mechanics. With regard to spatial discretization, attention will be focused on the movement from traditional "node to surface" methodologies for description of contact interaction, to new "surface to surface" algorithms that in most cases have their motivation in the mortar method. While an anticipated result of this evolution was the increased numerical accuracy produced by integral forms of the contact constraints, it has also been seen that considerable robustness in large sliding applications results from the non-local character of the formulation. In this discussion both of these advantages of the surface to surface framework will be demonstrated, as will recent extensions that enable reliable simulation of self-contact phenomena.

When extending computational contact formulations to the transient regime, the consideration of reliable time integrators for impact phenomena is of interest. Accordingly, we examine some of the issues associated with time stepping in semidiscrete formulations of contact/impact, with particular emphasis on the energy-momentum paradigm as applied to impact mechanics. We consider a form of the energy-momentum approach which encompasses dissipative phenomena (such as inelasticity and friction), and focus on a numerical approach that allows for velocity discontinuities to be incorporated into the contact updating scheme.

1 Problem Formulation

We begin by summarizing the equations governing the contact of solids, with extensive consideration of the continuum formulation of large deformation contact as described in such early sources as Laursen and Simo (1993) and Wriggers and Miehe (1994). We will consider the unilateral contact constraints between two deformable bodies here, with potential (Coulomb) frictional contact between them also encompassed by the framework.

We define open sets $\Omega^{(i)} \subset \mathbb{R}^{n_{sd}}, i = \{1, 2\}, n_{sd} = 2$ or 3, to denote reference configurations of two bodies. They have boundaries $\partial\Omega^{(i)}$ which are individually subdivided into non-intersecting regions $\Gamma_\sigma^{(i)}$ (Neumann boundary), $\Gamma_\varphi^{(i)}$ (Dirichlet boundary), and $\Gamma_c^{(i)}$ (contact boundary), each invariant with time and satisfying

$$\Gamma_\sigma^{(i)} \cup \Gamma_\varphi^{(i)} \cup \Gamma_c^{(i)} = \partial\Omega^{(i)}. \tag{1.1}$$

Given a time interval $\mathbb{I} = [0, T]$, and appropriate spaces for admissible deformations $\boldsymbol{\varphi}_t^{(i)}$ and admissible variational functions $\overset{*}{\boldsymbol{\varphi}}{}^{(i)}$, the weak form of the dynamic contact problem can be thus expressed for each body (i):

Find $\boldsymbol{\varphi}_t^{(i)}$ such that for all $\overset{*}{\boldsymbol{\varphi}}{}^{(i)}$:

$$\left\langle (\rho_0 \dot{\boldsymbol{V}}_t - \boldsymbol{f}_t) \ , \ \overset{*}{\boldsymbol{\varphi}} \right\rangle^{(i)} + \left\langle D\boldsymbol{\varphi}_t \boldsymbol{S}_t \ , \ \mathrm{G}RAD[\overset{*}{\boldsymbol{\varphi}}] \right\rangle^{(i)} - \left\langle \bar{\boldsymbol{t}}_t \ , \overset{*}{\boldsymbol{\varphi}} \right\rangle_{\Gamma_\sigma}^{(i)} = \left\langle \boldsymbol{t}_t \ , \overset{*}{\boldsymbol{\varphi}} \right\rangle_{\Gamma_c}^{(i)} \ . \tag{1.2}$$

In (1.2) and throughout we make use of a shorthand description of integral products,

$$\int_{\Omega^{(i)}} (\bullet) \cdot (\bullet) \, d\Omega^{(i)} := \langle \bullet, \bullet \rangle^{(i)}, \quad \text{and} \quad \int_{\Gamma^{(i)}} (\bullet) \cdot (\bullet) \, d\Gamma^{(i)} := \langle \bullet, \bullet \rangle_{\Gamma^{(i)}}^{(i)}. \tag{1.3}$$

Values in (1.2) include reference density, ρ_0; local material velocities, $\boldsymbol{V}_t = \dot{\boldsymbol{\varphi}}_t$; and a representation of the second (symmetric) Piola-Kirchhoff stress, denoted $\boldsymbol{S}_t$. The contact surface tractions ($\boldsymbol{t}_t$) are subject to a set of spatial geometric constraints dependent on the unknown deformation mappings $\boldsymbol{\varphi}_t^{(i)}$.

The variational form in (1.2) is composed of integral *v*irtual work expressions. Taking the left hand side and summing over the contacting bodies i gives the total virtual work of the combined non-contact forces on the system:

$$\begin{aligned} G(\boldsymbol{\varphi}_t, \overset{*}{\boldsymbol{\varphi}}) := \sum_i^2 \Bigg[& \left\langle (\rho_0 \dot{\boldsymbol{V}}_t - \boldsymbol{f}_t) \ , \ \overset{*}{\boldsymbol{\varphi}} \right\rangle^{(i)} \\ & + \left\langle D\boldsymbol{\varphi}_t \boldsymbol{S}_t \ , \ \mathrm{G}RAD[\overset{*}{\boldsymbol{\varphi}}] \right\rangle^{(i)} - \left\langle \bar{\boldsymbol{t}}_t \ , \overset{*}{\boldsymbol{\varphi}} \right\rangle_{\Gamma_\sigma}^{(i)} \Bigg]. \end{aligned} \tag{1.4}$$

We use a standard Lagrangian description for the contact surfaces, designating the material points as $\boldsymbol{X} \in \Gamma_c^{(1)}$ and $\boldsymbol{Y} \in \Gamma_c^{(2)}$, respectively. Contact points on $\Gamma_c^{(2)}$ are often mapped[1] from $\Gamma_c^{(1)}$ through a *c*losest point projection minimization

$$\bar{\boldsymbol{Y}}(\boldsymbol{X}, t) := \arg \min_{\boldsymbol{Y} \in \Gamma_c^{(2)}} \| \boldsymbol{\varphi}_t^{(1)}(\boldsymbol{X}) - \boldsymbol{\varphi}_t^{(2)}(\boldsymbol{Y}) \| \ . \tag{1.5}$$

Summing the right hand side of (1.2) and establishing force balance ($\boldsymbol{t}^{(1)} = -\boldsymbol{t}^{(2)} := \boldsymbol{t}$) along the shared contact surface ($\Gamma_c^{(1)} = \Gamma_c^{(2)} := \Gamma_c$), yields a single integral expression for the *v*irtual work of contact:

$$G_c(\boldsymbol{\varphi}_t, \overset{*}{\boldsymbol{\varphi}}) := -\left\langle \boldsymbol{t} \ , \ [\overset{*}{\boldsymbol{\varphi}}{}^{(1)}(\boldsymbol{X}) - \overset{*}{\boldsymbol{\varphi}}{}^{(2)}(\bar{\boldsymbol{Y}}(\boldsymbol{X}, t))] \right\rangle_{\Gamma_c} \tag{1.6}$$

The contact problem is thus compactly stated in virtual work terms:

[1] Note that this means of identification of contact points will be revised when the mortar contact framework is introduced

Find $\boldsymbol{\varphi}_t^{(i)}$, subject to the contact constraints, such that for all $\overset{*}{\boldsymbol{\varphi}}{}^{(i)}$:

$$G(\boldsymbol{\varphi}_t, \overset{*}{\boldsymbol{\varphi}}) + G_c(\boldsymbol{\varphi}_t, \overset{*}{\boldsymbol{\varphi}}) = 0 \tag{1.7}$$

In defining the local contact conditions in the continuum problem, we may use (1.5) to identify a unit direction vector $\boldsymbol{\nu}$ aligned with the contact surface normal, as well as a *g*ap function g, such that

$$\boldsymbol{\varphi}_t^{(1)}(\boldsymbol{X}) - \boldsymbol{\varphi}_t^{(2)}(\bar{\boldsymbol{Y}}(\boldsymbol{X}, t)) = -g\,\boldsymbol{\nu}; \tag{1.8}$$

and adopt the convention whereby $\boldsymbol{\nu}$ is directed *o*utward of $\Omega^{(2)}$ such that the 'gap' is *n*egative ($g < 0$) for admissible (i.e. non-penetrated) deformations. Manipulation of (1.8) defines a geometric description of the gap magnitude,

$$g = -\boldsymbol{\nu} \cdot \left(\boldsymbol{\varphi}_t^{(1)}(\boldsymbol{X}) - \boldsymbol{\varphi}_t^{(2)}(\bar{\boldsymbol{Y}}(\boldsymbol{X}, t)) \right). \tag{1.9}$$

Following the approach given in Laursen and Simo (1993), we parameterize the projection contact surface ($\Gamma_c^{(2)}$) in reference variables ξ^α, ($\alpha = 1, n_{sd} - 1$), and derive $n_{sd} - 1$ spatial vectors $\boldsymbol{\tau}_\alpha$ through differentiation of (1.8) within this parameterization, maintaining the closest-point minimization (indicated with the overbar notation) such that

$$\boldsymbol{\tau}_\alpha := \boldsymbol{\varphi}_t^{(2)}{}_{,\,\alpha}\left(\bar{\xi}(\boldsymbol{X}, t)\right) . \tag{1.10}$$

The tangential vectors $\boldsymbol{\tau}_\alpha$ are orthogonal to the surface normal $\boldsymbol{\nu}$, and are not in general orthonormal. This requires consideration of the associated metric and its inverse,

$$m_{\alpha\beta} := \boldsymbol{\tau}_\alpha \cdot \boldsymbol{\tau}_\beta \qquad [m^{\alpha\beta}] = [m_{\alpha\beta}]^{-1}, \tag{1.11}$$

in order to define the dual basis,

$$\boldsymbol{\tau}^\alpha := m^{\alpha\beta}\boldsymbol{\tau}_\beta . \tag{1.12}$$

(Note that here the summation convention is implied on repeated indices.) The contact forces, $\boldsymbol{t}$, can now be decomposed in terms of normal (t_N) and tangential ($t_{T\alpha}$) components, i.e.

$$\boldsymbol{t} = t_N\,\boldsymbol{\nu} - t_{T\alpha}\boldsymbol{\tau}^\alpha \tag{1.13}$$

Variations of the important surface quantities, namely the gap function g and the projected surface parameterization $\bar{\xi}$, can be generated as directional derivatives aligned with deformation variation $\overset{*}{\boldsymbol{\varphi}}$. Consider

$$\delta g = -\boldsymbol{\nu} \cdot [\overset{*}{\boldsymbol{\varphi}}{}^{(1)}(\boldsymbol{X}) - \overset{*}{\boldsymbol{\varphi}}{}^{(2)}(\bar{\boldsymbol{Y}}(\boldsymbol{X}, t))] \tag{1.14}$$

and

$$A_{\alpha\beta}\,\delta\bar{\xi}^\alpha = \boldsymbol{\tau}_\beta \cdot [\overset{*}{\boldsymbol{\varphi}}{}^{(1)}(\boldsymbol{X}) - \overset{*}{\boldsymbol{\varphi}}{}^{(2)}(\bar{\boldsymbol{Y}}(\boldsymbol{X}, t))] + g\,\boldsymbol{\nu} \cdot [-\overset{*}{\boldsymbol{\varphi}}{}^{(2)}{}_{,\beta}\left(\bar{\boldsymbol{Y}}(\boldsymbol{X}, t)\right)] , \tag{1.15}$$

where the symmetric matrix $A_{\alpha\beta}$ and its inverse $A^{\alpha\beta}$ are defined as

$$A_{\alpha\beta} := m_{\alpha\beta} + g\,\boldsymbol{\nu}\cdot[\overset{*}{\boldsymbol{\varphi}}{}^{(2)}{}_{,\alpha\beta}\,(\bar{\boldsymbol{Y}}(\boldsymbol{X},t))] \qquad \text{and} \qquad [A^{\alpha\beta}] := [A_{\alpha\beta}]^{-1}\ . \tag{1.16}$$

Temporal derivatives (denoted with a superimposed dot) are calculated in the same manner, yielding a local description for a *g*ap rate, $v_N = \dot{g}$ and local measures of relative tangential motion, or *s*lip rates, $v_T^\alpha = \dot{\bar{\xi}}^\alpha$ in terms of material velocities $\boldsymbol{V}^{(i)} = \dot{\boldsymbol{\varphi}}_t^{(i)}$:

$$v_N = \dot{g} = -\boldsymbol{\nu}\cdot[\boldsymbol{V}^{(1)}(\boldsymbol{X}) - \boldsymbol{V}^{(2)}(\bar{\boldsymbol{Y}}(\boldsymbol{X},t))], \tag{1.17}$$

and

$$A_{\alpha\beta}\,v_T^\alpha = A_{\alpha\beta}\,\dot{\bar{\xi}}^\alpha = \boldsymbol{\tau}_\beta\cdot[\boldsymbol{V}^{(1)}(\boldsymbol{X}) - \boldsymbol{V}^{(2)}(\bar{\boldsymbol{Y}}(\boldsymbol{X},t))] + g\,\boldsymbol{\nu}\cdot[-\boldsymbol{V}^{(2)}{}_{,\beta}\,(\bar{\boldsymbol{Y}}(\boldsymbol{X},t))]\ . \tag{1.18}$$

The slip rates can then be used in the definition of a relative and frame indifferent *s*lip velocity as proposed in Laursen and Simo (1993). Although many choices of reference frame are possible for posing of the frictional conditions, here we opt for a completely spatial definition of the slip velocity through use of the *s*patial metric $m_{\alpha\beta}$. Consider the definition

$$\boldsymbol{v}_T := \ v_T^\beta\boldsymbol{\tau}_\beta \ = \ m_{\alpha\beta}v_T^\beta\boldsymbol{\tau}^\alpha\ . \tag{1.19}$$

The descriptions in (1.14) and (1.15) are now combined with the contact force decomposition (1.13) and substituted into the variational equation to restate the virtual work of contact in terms of the surface variations,

$$G_c(\boldsymbol{\varphi}_t, \overset{*}{\boldsymbol{\varphi}}) = \int_{\Gamma_c}[t_N\delta g + t_{T\alpha}\delta\bar{\xi}^\alpha]\,d\Gamma_c. \tag{1.20}$$

The equivalence of (1.6) and (1.20) rests upon a pair of *c*omplementarity conditions,

$$t_N\,g = 0 \qquad \text{and} \qquad t_{T\alpha}\,g = 0, \tag{1.21}$$

which establish that the contact force magnitudes (non-zero only during contact) and the gap functions g (negative only when *o*ut of contact) cannot be mutually non-zero in the continuum description. The dilitational components of the tangential variation (the last term in each of (1.15) and (1.16)) can thus be considered as zero over the contact surfaces, validating the virtual work description (1.20).

With the global virtual work expression established, we wish now to apply a standard set of Kuhn-Tucker conditions in terms of the kinematic geometry, first in the normal direction, which remains the same for both frictionless and frictional contact:

$$\begin{aligned} g &\le 0 \\ t_N &\ge 0 \\ t_N\,g &= 0 \\ t_N\,v_N &= 0\ . \end{aligned} \tag{1.22}$$

In the event that frictional response is to be described, the contact conditions may be generically introduced via the following well-known equations of evolution:

$$\begin{aligned} \Phi(\boldsymbol{t}_T, t_N) &:= [t_{T_\alpha} m^{\alpha\beta} t_{T_\beta}]^{1/2} - \mu t_N \leq 0, \\ m_{\alpha\beta} v_T^\beta &= \dot{\gamma} \frac{t_{T_\alpha}}{[t_{T_\beta} m^{\beta\gamma} t_{T_\gamma}]^{1/2}}, \\ \dot{\gamma} &\geq 0, \\ \dot{\gamma}\Phi &= 0 \end{aligned} \tag{1.23}$$

where μ is the coefficient of friction.

2 Traditional Node-to-Surface Formulation of Contact

In developing a finite element representation of the system we have described, one begins by considering $\boldsymbol{\varphi}^{(i)^h}$ and $\overset{*}{\boldsymbol{\varphi}}{}^{(i)^h}$, finite-dimensional counterparts of $\boldsymbol{\varphi}^{(i)}$ and $\overset{*}{\boldsymbol{\varphi}}{}^{(i)}$. In particular, $\overset{*}{\boldsymbol{\varphi}}{}^{(i)^h} \in \mathcal{V}^{(i)^h} \subset \mathcal{V}^{(i)}$, while $\boldsymbol{\varphi}^{(i)^h}$, considered to be continuous in time, satisfies the following for each time t:

$$\boldsymbol{\varphi}_t^{(i)^h} \in \mathcal{C}_t^{(i)^h} \subset \mathcal{C}_t^{(i)}, \tag{2.1}$$

where $\mathcal{C}_t^{(i)}$ and $\mathcal{V}^{(i)}$ are the space of admissible configurations at time t and the space of admissible variations, respectively. Substitution of these finite dimensional quantities into the global variational principle (1.7) gives a set of nonlinear ordinary differential equations of the form

$$\boldsymbol{M}\ddot{\boldsymbol{d}}(t) + \boldsymbol{F}^{int}(\boldsymbol{d}(t)) + \boldsymbol{F}^{c}(\boldsymbol{d}(t)) = \boldsymbol{F}^{ext}(t), \tag{2.2}$$

subject to initial conditions on $\boldsymbol{d}$ and $\dot{\boldsymbol{d}}$. In (2.2), $\boldsymbol{M}$ is the mass matrix, $\boldsymbol{F}^{int}$ is the internal force vector, $\boldsymbol{F}^{c}$ is the contact force vector, and $\boldsymbol{F}^{ext}(t)$ is the external force vector (consisting of known data). The vector $\boldsymbol{d}$ symbolically represents the solution vector, or a vector of nodal values of the motion $\boldsymbol{\varphi}^h$. The manipulations necessary to derive $\boldsymbol{M}$, $\boldsymbol{F}^{int}$, and $\boldsymbol{F}^{ext}(t)$ from the virtual works of the contacting bodies have been extensively treated in the literature and will not be examined here. In the semidiscrete approach, approximate solutions to (2.2) are found by applying temporal integration schemes, as will be discussed later. The quasistatic equivalent of (2.2) is formally obtained by omission of the inertial term $\boldsymbol{M}\ddot{\boldsymbol{d}}$.

Equation (2.2) is in general highly nonlinear, mostly because of the terms $\boldsymbol{F}^{int}(\boldsymbol{d})$ and $\boldsymbol{F}^{c}(\boldsymbol{d})$. The first of these, the internal force vector, often contains both geometric and material nonlinearities, causing it to depend in a complex manner on $\boldsymbol{d}$. The second, the contact force vector, derives from expression (1.20) and has a form which depends on the method of spatial discretization. We will focus on this topic in the remainder of this section as well as the next. First we will consider node-to-surface methods for approximation of contact interaction, and then in the next section, a more recent surface-to-surface approach will be summarized.

2.1 Contact Surface Discretization

An important attribute of the contact formulation we consider is that all development depends only on the configurations and variations evaluated on the contact surfaces $\Gamma_c^{(i)}$, and *not* on values in the interiors of the bodies. Thus, in considering the discretization leading to specification of $\boldsymbol{F}^c(\boldsymbol{d})$, only the restrictions of $\boldsymbol{\varphi}^{(i)^h}$ and $\overset{*}{\boldsymbol{\varphi}}{}^{(i)}$ to $\Gamma_c^{(i)^h}$ need to be considered. These restrictions are considered to be collections of local mappings (denoted by superscript e's), defined over individual element surfaces.

For example, $\boldsymbol{\varphi}^{(1)^{h\,e}}(\boldsymbol{\eta})$, with $\boldsymbol{\eta} \in \mathcal{A}^{(1)^e}$, is expressed as

$$\boldsymbol{\varphi}^{(1)^{h\,e}}(\boldsymbol{\eta}) = \sum_{a=1}^{n_{nes}} N_a(\boldsymbol{\eta}) \boldsymbol{d}_a^{(1)}(t) \tag{2.3}$$

where $\boldsymbol{d}_a^{(1)}(t)$ is a nodal value of $\boldsymbol{\varphi}^{(1)^h}$, and n_{nes} is the number of nodes per element surface. $N_a(\boldsymbol{\eta})$ denotes a standard Lagrangian shape function, defined on the biunit square $\mathcal{A}^{(1)^e}$ for three dimensional problems and on $\mathcal{A}^{(1)^e} = [-1, 1]$ for two dimensional problems. The interpolation of $\overset{*}{\boldsymbol{\varphi}}{}^{(1)^h}$ is similarly conceived, via

$$\overset{*}{\boldsymbol{\varphi}}{}^{(1)^{h\,e}}(\boldsymbol{\eta}) = \sum_{a=1}^{n_{nes}} N^a(\boldsymbol{\eta}) \boldsymbol{c}_a^{(1)}, \tag{2.4}$$

where $\boldsymbol{c}_a^{(1)}$, a nodal value of $\overset{*}{\boldsymbol{\varphi}}{}^{(1)^h}$, is independent of time (and, owing to the arbitrary nature of $\overset{*}{\boldsymbol{\varphi}}{}^{(1)^h}$, will ultimately be argued to be arbitrary). Using the isoparametric interpolation scheme, one also has:

$$\boldsymbol{X}^{h^e}(\boldsymbol{\eta}) = \sum_{a=1}^{n_{nes}} N_a(\boldsymbol{\eta}) \boldsymbol{X}_a. \tag{2.5}$$

Analogues of (2.3)–(2.5) are assumed to hold for body (2); i.e.

$$\begin{aligned} \boldsymbol{\varphi}^{(2)^{h\,e}}(\boldsymbol{\xi}) &= \sum_{b=1}^{n_{nes}} N_b(\boldsymbol{\xi}) \boldsymbol{d}_b^{(2)}(t), \\ \overset{*}{\boldsymbol{\varphi}}{}^{(2)^{h\,e}}(\boldsymbol{\xi}) &= \sum_{b=1}^{n_{nes}} N_b(\boldsymbol{\xi}) \boldsymbol{c}_b^{(2)}, \end{aligned} \tag{2.6}$$

and

$$\boldsymbol{Y}^{h^e}(\boldsymbol{\xi}) = \sum_{b=1}^{n_{nes}} N_b(\boldsymbol{\xi}) \boldsymbol{Y}_b, \tag{2.7}$$

defined over element surface parent domains $\mathcal{A}^{(2)^e}$.

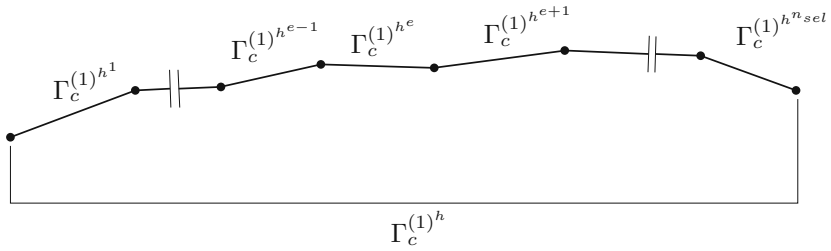

Figure 1. Division of the contact integral into subintegrals.

The contact virtual work in the discrete setting is now written, by substitution of the above discrete fields into (1.20), as

$$G^c(\boldsymbol{\varphi}_t^h, \overset{*}{\boldsymbol{\varphi}}{}^h) = \int_{\Gamma_c^{(1)^h}} [t_{N_t}^h \delta g^h + t_{T_{\alpha_t}}^h \delta \bar{\xi}^{\alpha^h}] \, d\Gamma, \tag{2.8}$$

where all quantities in the integrand depend on the discrete fields as given previously for the continuous case. As before, indices α, β, γ, etc. run between 1 and $n_{sd} - 1$ in (2.8) and in all other expressions in this chapter.

2.2 Numerical Integration of the Contact Integral

We now summarize the manner in which the contact virtual work integral is approximated in the traditional node-to-surface implementation. Dropping the subscript t's to reduce notation, (2.8) may be written as a sum of integrals over the n_{sel} element surfaces of $\Gamma_c^{(1)^h}$:

$$G^c(\boldsymbol{\varphi}^h, \overset{*}{\boldsymbol{\varphi}}{}^h) = \sum_{e=1}^{n_{sel}} \int_{\Gamma_c^{(1)^{h^e}}} [t_N^h \delta g^h + t_{T_\alpha}^h \delta \bar{\xi}^{\alpha^h}] \, d\Gamma. \tag{2.9}$$

where each subintegral of (2.9) is evaluated using quadrature. Figure 1 may be consulted for a graphic illustration of this division into subintegrals.

Performing a change of variables to the parent domain (i.e., $\mathcal{A}^{(1)^e}$) and applying an appropriate quadrature rule gives

$$\begin{aligned} G^c(\boldsymbol{\varphi}^h, \overset{*}{\boldsymbol{\varphi}}{}^h) &\approx \sum_{e=1}^{n_{sel}} \left\{ \sum_{k=1}^{n_{int}} W^k j(\boldsymbol{\eta}^k) [t_N^h(\boldsymbol{\eta}^k) \delta g^h(\boldsymbol{\eta}^k) + t_{T_\alpha}^h(\boldsymbol{\eta}^k) \delta \bar{\xi}^{\alpha^h}(\boldsymbol{\eta}^k)] \right\} \\ &= \sum_{e=1}^{n_{sel}} \left\{ \sum_{k=1}^{n_{int}} W^k j(\boldsymbol{\eta}^k) \delta \boldsymbol{\Phi}^{c^k} \cdot \boldsymbol{f}^{c^k} \right\} \end{aligned} \tag{2.10}$$