2

Modeling

Dynamic systems may be described in several ways. Physics often yields descriptions in terms of high order differential equations that involve input and output variables. Starting from this situation, a typical control theoretical problem is that of restating such input-output descriptions in terms of coupled, first-order differential equations, introducing new instrumental internal variables or states. For linear systems, it is possible to switch easily from input-output, or external, representation to state-space, or internal, representation, using the Laplace transform to change the domain of the representation. Although such a tool for symbolic computation is not available for nonlinear systems, we show, in this chapter, that internal, state-space representation can be derived from input-output descriptions (their construction will be described as the realization problem) and conversely, external, input-output descriptions can be derived from state-space descriptions (their construction will be described as the state elimination problem) in a nonlinear context, too.
To develop the tools required for dealing with this kind of problem, we will start by considering first the state elimination problem and, then, we will tackle the more relevant problem of constructing state-space representations from input/output relations.

2.1 State Elimination

Given the internal, or state-space, description of a system Σ, it is possible, in a sense to be made precise, to construct a representation of the relationship between input and output that it defines in a form that does not involve state variables. Although the validity of such a representation is only local, it nevertheless turns out to be useful for understanding the system behavior and, more important, its construction helps in clarifying the inverse problem of defining state variables and state equations from an input/output relation.
To describe the situation, we can consider, without additional difficulties, internal representations more general than (**??**), *i.e.*, representations of the

form

$$\begin{cases} \dot{x} = f(x, u, \ldots, u^{(s)}) \\ y = h(x, u, \ldots, u^{(s)}) \end{cases} \tag{2.1}$$

where, as usual, $x \in I\!R^n$, $u \in I\!R^m$, and $y \in I\!R^p$, and the entries of f and h, which depend also on a finite number of time derivatives of the input, are analytic functions.

So, given a system Σ of the form (2.1), the problem is to find, if possible, a set of input-output differential equations of the form

$$F_i(y, \dot{y}, \ldots, y^{(k)}, u, \dot{u}, \ldots, u^{(\gamma)}) = 0 \quad , \quad i = 1, \ldots, p \tag{2.2}$$

which admits as solution any pair $(y(t), u(t))$ such that $(y(t), u(t), x(t))$ is a solution, for some $x(t)$, of (2.1). Such a set of differential equations, if any exists, will be called an external, or input-output, representation of the system Σ described by (2.1).

Theorem 2.1. *Given a system Σ of the form (2.1), where the entries of f and h are analytic functions, there exist an integer γ and an open dense subset V of $I\!R^{n+m\gamma}$ such that, in the neighborhood of any point of V, there exists an input-output representation of the system of the form (2.2).*

Proof. The first step in constructing an input-output representation consists of applying a suitable change of coordinates. To this aim, let us denote by s_1 the minimum nonnegative integer such that

$$\text{rank}\frac{\partial(h_1, \ldots, h_1^{(s_1-1)})}{\partial x} = \text{rank}\frac{\partial(h_1, \ldots, h_1^{(s_1)})}{\partial x}$$

If $\partial h_1/\partial x \equiv 0$ we define $s_1 = 0$. Analogously for $1 < j \leq p$, let us denote by s_j the minimum integer such that

$$\begin{aligned} &\text{rank}\frac{\partial(h_1, \ldots, h_1^{(s_1-1)}; \ldots; h_j, \ldots, h_j^{(s_j-1)})}{\partial x} \\ &= \text{rank}\frac{\partial(h_1, \ldots, h_1^{(s_1-1)}; \ldots; h_j, \ldots, h_j^{(s_j)})}{\partial x} \end{aligned}$$

If

$$\text{rank}\frac{\partial(h_1, \ldots, h_{j-1}^{(s_{j-1}-1)})}{\partial x} = \text{rank}\frac{\partial(h_1, \ldots, h_{j-1}^{(s_{j-1}-1)}, h_j)}{\partial x}$$

we define $s_j = 0$. Write $K = s_1 + \ldots + s_p$. The vector

$$S = (h_1, \ldots, h_1^{s_1-1}, \ldots, h_p, \ldots, h_p^{s_p-1})$$

where h_j does not appear if $s_j = 0$, satisfies the following relation

$$\text{rank}\left[\frac{\partial S}{\partial x}\right] = K, \text{ for almost every } x$$

It will be established in Chapter 4 that the case $K < n$ corresponds to nonobservable systems. In this case, there exist analytic functions $g_1(x), \ldots, g_{n-K}(x)$ such that the matrix

$$J = \frac{\partial(S, g_1, \ldots, g_{n-K})}{\partial x}$$

has full rank n. Then the system of equations

$$\left\{\begin{aligned} \tilde{x}_1 &= h_1(x, u, \ldots, u^{(\alpha)}) \\ &\vdots \\ \tilde{x}_{s_1} &= h_1^{(s_1-1)}(x, u, \ldots, u^{(\alpha+s_1-1)}) \\ \tilde{x}_{s_1+1} &= h_2(x, u, \ldots, u^{(\alpha)}) \\ &\vdots \\ \tilde{x}_{s_1+s_2} &= h_2^{(s_2-1)}(x, u, \ldots, u^{(\alpha+s_2-1)}) \\ &\vdots \\ \tilde{x}_{s_1+s_2+\ldots+s_p} &= h_p^{(s_p-1)}(x, u, \ldots, u^{(\alpha+s_p-1)}) \\ \tilde{x}_{s_1+s_2+\ldots+s_p+i} &= g_i(x, u, \ldots, u^{(\gamma)}) \qquad i = 1, \ldots, n-K \end{aligned}\right. \tag{2.3}$$

is of the form $F_i(x, \tilde{x}, u, \ldots, u^{(\gamma)}) = 0, i = 1, \ldots, n$ with

$$\partial(F_1, \ldots, F_n)/\partial(x_1, \ldots, x_n) = J$$

To avoid the introduction of new notations, it is not restrictive to assume $\gamma \geq \max\{\alpha + s_i - 1, i = 1, \ldots, p\}$. The determinant of J is an analytic function whose set of zeros has an empty interior, so there exists an open dense subset V of $\mathbb{R}^{n+m\gamma}$ such that $\det J$ is different from zero at every point of V and the implicit function theorem applies. Therefore there exist n functions

$$x_i = \phi_i(\tilde{x}, u, \ldots, u^{(\gamma)}) \quad \text{for} \quad 1 \leq i \leq n$$

which define a local diffeomorphism ϕ parametrized by $u, \ldots, u^{(\gamma)}$:

$$x = \phi(\tilde{x}) \tag{2.4}$$

By applying the change of coordinates induced by (2.4), the system (2.1) becomes

$$\left\{\begin{aligned}
\dot{\tilde{x}}_1 &= \tilde{x}_2 \\
\dot{\tilde{x}}_2 &= \tilde{x}_4 \\
&\vdots \\
\dot{\tilde{x}}_{s_1} &= h_1^{(s_1)}(\phi(\tilde{x}), u, \dots, u^{(\gamma)}) \\
\dot{\tilde{x}}_{s_1+1} &= \tilde{x}_{s_1+2} \\
&\vdots \\
\dot{\tilde{x}}_{s_1+s_2} &= h_2^{(s_2)}(\phi(\tilde{x}), u, \dots, u^{(\gamma)}) \\
&\vdots \\
\dot{\tilde{x}}_{s_1+\dots+s_p} &= h_p^{(s_p)}(\phi(\tilde{x}), u, \dots, u^{(\gamma)}) \\
\dot{\tilde{x}}_{s_1+\dots+s_p+i} &= g_i(\tilde{x}), u, \dots, u^{(\gamma)}) \qquad i = 1, \dots, n-K \\
y_1 &= \tilde{x}_1 \\
y_2 &= \tilde{x}_{s_1+1} \\
&\vdots \\
y_p &= \tilde{x}_{s_1+\dots+s_{p-1}+1}
\end{aligned}\right. \tag{2.5}$$

In the neighborhood of any point where $\det J \neq 0$, also

$$\frac{\partial h_i^{(s_i)}}{\partial x} \in \operatorname{span}_{\mathcal{K}} \left\{ \frac{\partial h_1}{\partial x}, \dots, \frac{\partial h_1^{(s_1-1)}}{\partial x}, \frac{\partial h_2}{\partial x}, \dots, \frac{\partial h_i^{(s_i-1)}}{\partial x} \right\}$$

so that

$$\begin{aligned}
\frac{\partial h_i^{(s_i)}}{\partial \tilde{x}} &= [c_1 \ \dots \ c_{s_1+\dots+s_i} \ 0 \ \dots \ 0] J \frac{\partial x}{\partial \tilde{x}_j} \\
&= [c_1 \ \dots \ c_{s_1+\dots+s_i} \ 0 \ \dots \ 0] \, e_j = 0 \qquad j > s_1 + \dots + s_i
\end{aligned}$$

where e_j is the jth column of the identity matrix. Therefore the functions $h_i^{(s_i)}(\phi(\tilde{x}), u, \dots, u^{(\gamma)})$ depend only on $\tilde{x}_1, \dots, \tilde{x}_{s_1+\dots+s_i}$.
Since the following identities hold,

$$\begin{aligned}
&y_1 = \tilde{x}_1, \\
&\dot{y}_1 = \tilde{x}_2, \dots, \\
&y_1^{(r)} = \tilde{x}_{1+r} \ \text{ for } r = 0, \dots, s_1 - 1 \\
&\vdots \\
&y_j = \tilde{x}_{s_1+\dots+s_{j-1}+1} \\
&\dot{y}_j = \tilde{x}_{s_1+\dots+s_{j-1}+2}, \dots, \\
&y_j^{(r)} = \tilde{x}_{s_1+\dots+s_{j-1}+1+r} \ \text{ for } r = 0, \dots, s_j - 1, \ j = 2, \dots, p
\end{aligned}$$

From (2.5), we get the input-output relations we were looking for:

$$\begin{aligned}
y_1^{(s_1)} &= h_1^{(s_1)}(\phi(y_1, \dot{y}_1, \ldots, y_1^{(s_1-1)}), u, \ldots, u^{(\gamma)}) \\
&\vdots \\
y_j^{(s_j)} &= h_j^{(s_j)}(\phi(y_1, \ldots, y_1^{(s_1-1)}, y_j, \ldots, y_j^{(s_j-1)}), u, \ldots, u^{(\gamma)}) \\
&\vdots \\
y_p^{(s_p)} &= h_p^{(s_p)}(\phi(y_1, \ldots, y_1^{(s_1-1)}, \ldots, y_p, \ldots, y_p^{(s_p-1)}), u, \ldots, u^{(\gamma)})
\end{aligned} \tag{2.6}$$

∎

The input-output equations (2.6) are not uniquely defined since, for instance, if K is less than n, different choices of the functions $g_i(x, u, \ldots, u^{(\gamma)})$ produce a different system (2.3).

Instead of $\{s_1, \ldots, s_p\}$, it is possible to use the observability indices as defined in Chapter 4 to derive an analogous input-output equation.

2.2 Examples

Example 2.2. For the system

$$\begin{cases} \dot{x}_1 = x_3 u_1 \\ \dot{x}_2 = u_1 \\ \dot{x}_3 = u_2 \\ y_1 = x_1 \\ y_2 = x_2 \end{cases}$$

we have $\dot{y}_1 = x_3 u_1$, $\ddot{y}_1 = u_2 u_1 + x_3 \dot{u}_1$, and finally

$$\ddot{y}_1 = u_2 u_1 + (\dot{y}_1 / u_1)\dot{u}_1$$

The last equation holds at every point in which $u_1 \neq 0$. For the second output, $\dot{y}_2 = u_1$ immediately.

The following example shows that for a more general nonlinear system, where $\dot{x}$ does not appear explicitly, such as

$$F(x, \dot{x}, u, \ldots, u^{(\nu)}) \tag{2.7}$$

the method described above cannot be applied.

Example 2.3. Consider the system

$$\begin{cases} (\dot{x} - u)^2 = 0 \\ \qquad\quad y = x \end{cases}$$

The implicit function theorem cannot be invoked to obtain x, since for every x and every u, $\partial(\dot{x} - u)^2/\partial x = 0$. By the way, an input-output relation for this example is given by

$$(\dot{y} - u)^2 = 0$$

or by

$$(\dot{y} - u) = 0$$

Results similar to those described above may be found in [156]. A state elimination method which yields global results is studied in [44].

2.3 Generalized Realization

Let us consider now the problem of going from an external, input-output representation of a dynamic system to an internal, state-space representation, that, in a sense to be made precise, defines the same relation between inputs and outputs as the external representation. We are interested in what is called the realization problem. In general, as an outcome of the realization process, starting from an external representation, one would like to obtain internal, state-space representations of the form (1.4) or, at least, of the form (1.5). These will be termed *classical realizations*; more general representations, as those we will discuss in this section, will be termed *generalized realization.*
To begin with, we first recall some results from [54], which follow quite naturally from elementary manipulation of input-output equations and which yield a generalized realization. In the next section, a necessary and sufficient condition is given for the existence of a classical realization in the single-input/single-output (SISO) case.
Consider an input-output differential equation of the form

$$F(y, \ldots, y^{(k)}, u, \ldots, u^{(s)}) = 0 \tag{2.8}$$

where u and y are, respectively, a scalar input and a scalar output, F is a meromorphic function of its arguments; and $\frac{\partial F}{\partial y^{(k)}}$ is generically nonzero. An internal representation of the system described by (2.8) can easily be constructed by introducing the new variable $x = (x_1, \ldots, x_k)$, defined by

$$(x_1, \ldots, x_k) = (y, \ldots, y^{(k-1)}) \tag{2.9}$$

This yields the following set of implicit state equations

$$\left\{ \begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{k-1} &= x_k \\ F(x_1, \ldots, x_k, \dot{x}_k, u, \ldots, u^{(s)}) &= 0 \end{aligned} \right. \tag{2.10}$$

The assumption about $\frac{\partial F}{\partial y^{(k)}}$ and the implicit function theorem , now, allow us to write, at least locally,

$$\left\{ \begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{k-1} &= x_k \\ \dot{x}_k &= \varphi(x, u, \dot{u}, \ldots, u^{(s)}) \\ y &= x_1 \end{aligned} \right. \tag{2.11}$$

Equations (2.11) give a representation of the input-output relation described by (2.8) with internal variables. Compared with (1.4), the representation given by (2.11) can be said to be a generalized realization; the adjective "generalized" accounts for the presence of derivatives of u. According to this, the variable x can be interpreted as a generalized state variable. In addition, note that the application of the implicit function theorem, beside being nonconstructive, does not guarantee that φ is a meromorphic function.

Example 2.4. Consider the input-output equation $\dot{y}^2 = y + u$. The above procedure yields the implicit state equations

$$\dot{x}^2 = x + u$$

or, locally, one of the following explicit realizations, depending on whether $\dot{y} > 0$ or $\dot{y} < 0$.

$$\begin{cases} \dot{x} = \sqrt{x+u} \\ y = x \end{cases} \tag{2.12}$$

$$\begin{cases} \dot{x} = -\sqrt{x+u} \\ y = x \end{cases} \tag{2.13}$$

Note that the right-hand side of the state equations in the above representations is not meromorphic at the origin.

In general, the above procedure does not yield classical realizations of the form (1.4). Also linear input-output relations, in case transmission zeros are present, give rise, in this way, to generalized realizations. It can be said that, in general, the presence of derivatives of u is somehow related to the presence of zero dynamics (this concept will be made more precise in Section 5.6, see also [88]) However, as we will show in the next section, generalized realizations of the form (2.11) may be transformed under suitable hypotheses into a realization containing no derivatives of u.

Example 2.5. Consider the linear input-output relation $\ddot{y} = u + \dot{u}$ that corresponds to the transfer function $\frac{s+1}{s^2}$, having a zero in $s = -1$. Although the input-output relation is linear, the above procedure yields a generalized realization:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -u - \dot{u} \\ y = x_1 \end{cases}$$

The notions of controllability/accessibility and of observability that one can use in characterizing the structure of internal representations are reported in Chapters 3 and 4. Without going into the details now, we mention that, with respect to those notions, realization (2.11) is in general observable, but not necessarily accessible . In this sense, it is not minimal.

2.4 Classical Realization

Conditions for ensuring the existence of a classical realization of the form (1.4), in particular containing no derivatives of the input, are fully characterized in [33] (see also [28, 47, 56, 91, 92, 150, 151, 158]). There exist simple input-output relations which do not admit a classical realization. A typical example in this sense is given by the input-output relation $\ddot{y} = \dot{u}^2$.
Here we introduce an elementary result that fully solves the problem for the input-output relations having the particular form

$$y^{(k)} = \varphi(y, \dot{y}, \ldots, y^{(k-1)}, u, \dot{u}, \ldots, u^{(s)}). \tag{2.14}$$

where φ is a meromorphic function of its arguments and $\frac{\partial \varphi}{\partial y^{(k)}}$ is generically nonzero. The input-output relation (2.14) admits a realization if and only if the right-hand side of (4.17) has a special polynomial structure in the derivatives of u. To investigate this structure, consider the dynamic system Σ_e whose input is $u^{(s+1)}$ and whose state is $(y, \dot{y}, \ldots, y^{(k-1)}, u, \dot{u}, \ldots, u^{(s)})$.

$$\Sigma_e : \frac{d}{dt}\begin{bmatrix} y \\ \vdots \\ y^{(k-2)} \\ y^{(k-1)} \\ u \\ \vdots \\ u^{(s-1)} \\ u^{(s)} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \vdots \\ y^{(k-1)} \\ \varphi \\ \dot{u} \\ \vdots \\ u^{(s)} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u^{(s+1)} \tag{2.15}$$
$$= f_e + g_e u^{(s+1)}$$

Given system (2.14), define the field $\mathcal{K}$ of meromorphic functions in a finite number of variables y, u, and their time derivatives. Let $\mathcal{E}$ be the formal vector space $\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi \mid \varphi \in \mathcal{K}\}$.

Define the following subspace of $\mathcal{E}$

$$\mathcal{H}_1 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y}, \ldots, \mathrm{d}y^{(k-1)}, \mathrm{d}u, \ldots, \mathrm{d}u^{(s)}\}$$

Obviously, any one-form in $\mathcal{H}_1$ has to be differentiated at least once to depend explicitly on $\mathrm{d}u^{(s+1)}$. Let $\mathcal{H}_2$ denote the subspace of $\mathcal{E}$ which consists of all one-forms that have to be differentiated at least twice to depend explicitly on $\mathrm{d}u^{(s+1)}$. From (2.15), one easily computes

$$\mathcal{H}_2 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y}, \ldots, \mathrm{d}y^{(k-1)}, \mathrm{d}u, \ldots, \mathrm{d}u^{(s-1)}\}$$

$\mathcal{H}_2$ is a subspace of $\mathcal{H}_1$ which is more generally computed as

$$\mathcal{H}_2 = \mathrm{span}_{\mathcal{K}}\{\omega \in \mathcal{H}_1 \mid \dot{\omega} \in \mathcal{H}_1\} \tag{2.16}$$

More generally, define $\mathcal{H}_i$ as the subspace of $\mathcal{E}$ which consists of all one-forms that have to be differentiated at least i times to depend explicitly on $\mathrm{d}u^{(s+1)}$.

More precisely, the subspaces $\mathcal{H}_i$ are defined by induction as follows for $i \geq 2$.

$$\mathcal{H}_{i+1} = \mathrm{span}_{\mathcal{K}}\{\omega \in \mathcal{H}_i \mid \dot{\omega} \in \mathcal{H}_i\}$$

These subspaces will be used extensively later in this book and especially in Chapter 3.

2.5 Input-output Equivalence and Realizations

To introduce the equivalence of input-output systems and to study their minimal state-space realization, we will use systems Σ_e, as defined in (2.15). Consider $\mathcal{H}_\infty^* = \mathrm{span}_{\mathcal{K}}\{\omega \in \mathcal{H}_1^* \mid \omega^{(k)} \in \mathcal{H}_1^*, \forall k \geq 0\} = 0$. Each nonzero vector in $\mathcal{H}_\infty^*$ is said to be *autonomous* for system (2.14).

2.5.1 Irreducible Input-output Systems

In this section, we will formalize a *reduction* algorithm to obtain the notion of input-output equivalence and a definition of realization.

Definition 2.6 (Irreducible input-output system). *System (2.14) is said to be an* irreducible input-output system *if the associated system (2.15) satisfies*

$$\mathcal{H}_\infty = 0$$

Example 2.7. The input-output system $\ddot{y} = yu^2 + y\dot{u}$ is irreducible since

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y \\ \dot{y} \\ u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} \dot{y} \\ yu^2 + y\dot{u} \\ \dot{u} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \ddot{u}$$

is such that $\mathcal{H}_\infty = 0$. It is worth noting that the set of solutions $(u(t), y(t))$ of $\dot{y} = yu$ is a subset of the set of solutions of $\ddot{y} = yu^2 + y\dot{u}$, but the systems are not "equivalent" according to the forthcoming Definition 2.13.

Example 2.8. $\ddot{y} = \dot{u} + (\dot{y} - u)^2$ is not irreducible since

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y \\ \dot{y} \\ u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} \dot{y} \\ \dot{u} + (\dot{y} - u)^2 \\ \dot{u} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \ddot{u}$$

is not irreducible since $\mathrm{d}(\dot{y} - u) \in \mathcal{H}_\infty$ and we will claim that $\dot{y} = u$ is an irreducible input-output system of $\ddot{y} = \dot{u} + (\dot{y} - u)^2$.

2.5.2 Reduced Differential Form

We are interested in minimal realizations, *i.e.* of the lowest order. We introduce definitions of *reduced differential form, reduced input-output system and irreducible differential form, etc.* to reach that goal.

Definition 2.9 (Reduced differential form). *An exact form* $\mathrm{d}\phi' \in \mathcal{H}_\infty$ *is said to be a* reduced differential form *of system (2.14) if*

(a) $\mathrm{d}\phi' \not\equiv 0$
(b) $\mathrm{d}\phi' \in \mathcal{H}_\infty$.

Definition 2.10 (Reduced input-output system). *Let* $\mathrm{d}\phi'$ *be a reduced differential form, that produces the differential equation*

$$\phi'(y, \cdots, y^{(k'-1)}, y^{(k')}, u, \cdots, u^{(s')}) = 0 \tag{2.17}$$

such that $\partial\phi'/\partial y^{(k')} \neq 0$, $\partial\phi'/\partial u^{(s')} \neq 0$, *and* $\partial^2\phi'/\partial {y^{(k')}}^2 \equiv 0$ *with* $k' > 0$, $s' \geq 0$. *Equation (2.17) has a unique solution under the condition* $\partial\phi'/\partial y^{(k')} \not\equiv 0$

$$y^{(k')} = \varphi'(y, \cdots, y^{(k'-1)}, u, \cdots, u^{(s')}) \tag{2.18}$$

Then (2.18) is called a reduced input-output system *of system (2.14).*

Definition 2.11 (Irreducible differential form). *If (2.18) is an irreducible input-output system in the sense of Definition 2.6, then* $\mathrm{d}(y^{(k')} - \varphi')$ *is said to be an irreducible differential form of (2.14).*

Example 2.12 (Example 2.8 cont'd). $\mathrm{d}(\dot{y}-u) \in \mathcal{H}_\infty$ and $\dot{y} = u$ is an irreducible system. Thus, $\phi' = \dot{y} - u = 0$ is an irreducible input-output system of $\ddot{y} = \dot{u} + (\dot{y} - u)^2$.

It is not true that any input-output system has an irreducible input-output system. Consider

$$\ddot{y} = \frac{\dot{y}\dot{u}}{u} \tag{2.19}$$

$d\phi' = \mathrm{d}(\dot{y}/u)$ is a reduced differential form of (2.19) according to Definition 2.9. Thus, system (2.19) is not irreducible. Let $\phi' = \dot{y}/u = 0$, which is not an *irreducible input-output system* in the sense of the above Definition. Therefore, system (2.19) does not admit any irreducible input-output system.

In the special case of linear time-invariant systems, the reduction procedure corresponds to a pole/zero cancellation in the transfer function. For nonlinear systems, the above procedure also generalizes the so-called *primitive step* in [28].

2.5.3 Input-output Equivalence

We restrict our attention to the family of input-output systems that admit an irreducible input-output system: see Definition 2.6. Therefore, it is possible to introduce an equivalence relation on the family.

Definition 2.13 (Input-output equivalence). *Two input-output systems are said to be input-output equivalent if they have the same irreducible input-output system representation*

$$y^{(\kappa)} = \varphi(y, \dots, y^{(\kappa-1)}, u, \dots, u^{(\sigma)}) \tag{2.20}$$

Example 2.14. The two systems

$$\ddot{y} = \dot{u} - 2(\dot{y} - u)^2$$

and

$$y^{(3)} = \ddot{u}$$

do admit the same irreducible input-output system, $\dot{y} = u$.

2.5.4 Realizations

A general definition of *realization* is given, that describes the relationships between state-space equations (1.1) and input-output equations (2.14).

An algorithm realizing the state-space systems (1.1) from input-output systems (2.14) will be provided in Section 2.8.1, as well as a necessary and sufficient condition for the existence of such a realization.

Definition 2.15 (Realization). *A state-space system (1.1) is said to be a realization of the input-output system (2.14) if the elimination of the state variables in (1.1) yields an input-output equation described by*

$$y^{(\kappa)} = \phi(y, \dots, y^{(\kappa-1)}, u, \dots, u^{(\sigma)})$$

which is input-output equivalent to (2.14).

The system (2.14) is said to be realizable if there exists a realization in the sense of Definition 2.15.

2.6 A Necessary and Sufficient Condition for the Existence of a Realization

We make use here of the subspaces introduced in (2.16), to derive a full characterization of the existence of a classical realization.

Theorem 2.16. *There exists an observable state-space system*

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x) \end{cases} \tag{2.21}$$

which is a realization for (2.14) if and only if

- $k > s$
- *and,* $\mathcal{H}_i$ *is integrable for each* $i = 1, \ldots, s+2$.

Proof. *Sufficiency:* Let $\{\mathrm{d}\xi_1, \ldots, \mathrm{d}\xi_k\}$ be a basis of $\mathcal{H}_{s+2}$. From the construction of the subspaces $\mathcal{H}_i$,

$$\begin{aligned} \mathcal{H}_{s+1} &= \mathcal{H}_{s+2} \oplus \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u\} \\ \mathcal{H}_{s} &= \mathcal{H}_{s+2} \oplus \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}\dot{u}\} \\ &\vdots \\ \mathcal{H}_{1} &= \mathcal{H}_{s+2} \oplus \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u, \ldots, \mathrm{d}u^{(s)}\} \end{aligned} \tag{2.22}$$

Introduce the following coordinate transformation for the system (2.15):

$$\begin{aligned} x_1 &= \xi_1(y, \dot{y}, \ldots, u^{(s)}) \\ &\vdots \\ x_k &= \xi_k(y, \dot{y}, \ldots, u^{(s)}) \\ x_{k+1} &= u \\ &\vdots \\ x_{k+s+1} &= u^{(s)} \end{aligned} \tag{2.23}$$

From $\mathcal{H}_{s+2} \subset \mathcal{H}_{s+1}$, it follows $\mathrm{d}\dot{\xi}_i = \sum_{j=1}^{k} \alpha \mathrm{d}\xi + \beta \mathrm{d}u$, for each $j = 1, \ldots, k$. Let $x = (x_1, \ldots, x_k)$. Thus, at least locally,

$$\dot{x} = f(x,u) \tag{2.24}$$

The assumption $k > s$ indicates that the output y depends only on x.
Necessity: Assume that the observable state-space system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned}$$

is a realization for the input-output system (2.14). Since the state-space system is proper, necessarily $k > s$.

$$\begin{aligned} \mathcal{H}_1 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u, \ldots, \mathrm{d}u^{(s)}\} \\ &\vdots \\ \mathcal{H}_{s+1} &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u\} \\ \mathcal{H}_{s+2} &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\} \end{aligned}$$

From (2.23), the spaces $\mathcal{H}_i$ are integrable as expected. ∎

Example 2.17. Let $\ddot{y} = \dot{u}^2$, and compute

$$\begin{aligned}\mathcal{H}_1 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y}, \mathrm{d}u, \mathrm{d}\dot{u}\}\\ \mathcal{H}_2 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y}, \mathrm{d}u\}\\ \mathcal{H}_3 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y} - 2\dot{u}\mathrm{d}u)\}\end{aligned}$$

Since $\mathcal{H}_3$ is not integrable, there does not exist any state-space system generating $\ddot{y} = \dot{u}^2$. This can be checked directly, or using some results in [33].

Example 2.18. Let $\ddot{y} = u^2$. The conditions of Theorem 2.16 are fulfilled and the state variables $x_1 = y$ and $x_2 = \dot{y}$ yield

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u^2\\ y &= x_1\end{aligned}$$

whose state elimination yields $\ddot{y} = u^2$.

2.7 Minimal Realizations

The notion of minimality here is standard for linear systems and means that the dimension of the state-space system equals the order of some reduced transfer function.

A minimal realization can be obtained directly from the input-output equation. The notion of irreducible form is used as it is for linear time-invariant systems. A minimal realization is obtained when constructing a realization as in the proof of Theorem 2.16, or applying the algorithm in Section 2.8.1, to an irreducible input-output system, whenever it exists. More precisely, one has

Theorem 2.19. *Given an input-output system (2.14), assume that the conditions in Theorem 2.16 are fulfilled. Then, there exists an observable and controllable, i.e., minimal, realization of order k for (2.14), if and only if (2.14) is an irreducible input-output system.*

Proof. Given (2.14), the generating system (2.21) obtained from Theorem 2.16 is observable. The extended system (2.15) can be written in the coordinates (2.23). It then reads as the composite system of system (2.24) and the controllable string of integrators $\dot{u}^{(i)} = u^{(i+1)}$, $i = 0, ..., s$. Thus, (2.15) is accessible if and only if (2.24) is controllable. The result of Theorem 2.19 follows since (2.15) is controllable if and only if (2.14) is irreducible, by Definition 2.6. ∎

Example 2.20. Consider

$$\phi = \ddot{y} - \dot{y}u - y\dot{u}.$$

Compute $f_e = \begin{pmatrix} \dot{y} \\ \dot{y}u + y\dot{u} \\ \dot{u} \\ 0 \end{pmatrix}$ and $g_e = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Thus, $\mathrm{d}\phi_r = \mathrm{d}(\dot{y} - yu) \in \mathcal{H}_\infty$. An irreducible differential form of $\phi = 0$ is $\mathrm{d}\phi_r = \mathrm{d}(\dot{y} - yu)$. A minimal realization is obtained for $\phi = 0$ as

$$\begin{cases} \dot{x} = xu \\ y = x \end{cases}$$

2.8 Affine Realizations

2.8.1 A Realization Algorithm

Under the conditions of Theorem 2.16, any basis of $\mathcal{H}_{s+2}$ defines a state space of the input-output system (2.14). The purpose of this section is to give an algorithmic construction of a canonical affine state-space representation; it results from a special choice of the basis for $\mathcal{H}_{s+2}$ under some special structure of the input-output equation. Consider the input-output equation (2.14).

Algorithm 2.21

Step 1.
Let $r := k - s$, *then* $\{\mathrm{d}y, \dots, \mathrm{d}y^{(r-1)}\}$ *is a basis for*

$$\mathcal{X}_1 := \mathcal{H}_{s+2} \cap \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y^{(j)}, j \geq 0\}$$

- *If* $\partial^2\varphi/\partial(u^{(s)})^2 \neq 0$, *stop!*
- *If* $\partial^2\varphi/\partial(u^{(s)})^2 = 0$ *and* $\mathrm{d}(\partial y^{(k)}/\partial u^{(s)}) \neq 0$, *define*

$$y_{11} = \partial y^{(k)}/\partial u^{(s)} \tag{2.25}$$

If $\mathrm{d}(y^{(r)} - \frac{\partial y^{(k)}}{\partial u^{(s)}}u) \neq 0$, *define*

$$y_{12} = y^{(r)} - \frac{\partial y^{(k)}}{\partial u^{(s)}}u \tag{2.26}$$

y_{11} *and* y_{12} *are called* auxiliary *outputs.*

Step 2.
- *If* $\mathcal{H}_{s+2} \cap \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_{11}^{(i)}, i \geq 0\} = 0$, *then stop!*
- *Let* $\{\mathrm{d}y, \dots, \mathrm{d}y^{(r-1)}; \mathrm{d}y_{11}, \dots, \mathrm{d}y_{11}^{(r_{11}-1)}\}$ *be a basis for*

$$\mathcal{X}_{21} := \mathcal{X}_1 + \mathcal{H}_{s+2} \cap \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_{11}^{(i)}, i \geq 0\}$$

where $r_{11} = \dim\mathcal{X}_{21} - \dim\mathcal{X}_1$.
- *If* $\mathcal{H}_{s+2} \cap \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_{12}^{(i)}, i \geq 0\} = 0$, *then stop!*
- *Let* $\{\mathrm{d}y, \dots, \mathrm{d}y_{11}^{(r_{11}-1)}; \mathrm{d}y_{12}, \dots, \mathrm{d}y_{12}^{(r_{12}-1)}\}$ *be a basis for*

$$\mathcal{X}_2 := \mathcal{X}_{21} + \mathcal{H}_{s+2} \cap \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_{12}^{(i)}, i \geq 0\}$$

where $r_{12} = \dim\mathcal{X}_2 - \dim\mathcal{X}_{21}$.

- *If* $\forall \ell \geq r_{1j}$, $\mathrm{d}y_{1j}^{(\ell)} \in \mathcal{X}_2$, *set* $s_{1j} = -1$, *for* $j = 1, 2$.

If $\exists \ell \geq r_{1j}, \mathrm{d}y_{1j}^{(\ell)} \notin \mathcal{X}_2$, *then define* $s_{1j} \geq 0$ *as the smallest integer such that, abusing the notation, one has locally*

$$y_{1j}^{(r_{1j}+s_{1j})} = y_{1j}^{(r_{1j}+s_{1j})}(y^{(\lambda)}, y_{11}^{(\sigma_{11})}, y_{12}^{(\sigma_{12})}, u, \ldots, u^{(s_{1j})})$$

where $0 \leq \lambda < r$, $0 \leq \sigma_{11} < r_{11} + s_{11}$, $0 \leq \sigma_{12} < r_{12} + s_{12}$.

- *If* $s_{11} \geq 0$ *and* $\partial^2 y_{11}^{(r_{11}+s_{11})}/\partial(u^{(s_{11})})^2 \neq 0$

or if $s_{12} \geq 0$ *and* $\partial^2 y_{12}^{(r_{12}+s_{12})}/\partial(u^{(s_{12})})^2 \neq 0$ *stop!*

- *If* $\mathcal{X}_2 + \mathcal{U} = \mathcal{Y} + \mathcal{U}$, *and* $\partial^2 y_{1j}^{(r_{1j}+s_{1j})}/\partial(u^{(s_{1j})})^2 = 0$ *whenever* $s_{1j} \geq 0$, *then the algorithm stops and the realization is complete. Otherwise, define the new* auxiliary *outputs, whenever* $\mathrm{d}(\partial y_{1j}^{(r_{1j}+s_{1j})}/\partial u^{(s_{1j})}) \neq 0$, *respectively,* $\mathrm{d}(y_{1j}^{(r_{1j})} - \dfrac{\partial y_{1j}^{(r_{1j}+s_{1j})}}{\partial u^{(s_{1j})}} u) \neq 0$*:*

$$\begin{aligned}
y_{21} &= \frac{\partial y_{11}^{(r_{11}+s_{11})}}{\partial u^{(s_{11})}} \\
y_{22} &= y_{11}^{(r_{11})} - \frac{\partial y_{11}^{(r_{11}+s_{11})}}{\partial u^{(s_{11})}} u \\
y_{23} &= \frac{\partial y_{12}^{(r_{12}+s_{12})}}{\partial u^{(s_{12})}} \\
y_{24} &= y_{12}^{(r_{12})} - \frac{\partial y_{12}^{(r_{12}+s_{12})}}{\partial u^{(s_{12})}} u
\end{aligned}$$

Step i+1.

From Step i, one has defined a set of numbers $r_{i-1,j}$ *and* $s_{i-1,j}$ *as well as the auxiliary outputs*

$$\begin{aligned}
y_{i,2j-1} &= \partial y_{i-1,j}^{(r_{i-1,j}+s_{i-1,j})}/\partial u^{(s_{i-1,j})} \\
y_{i,2j} &= y_{i-1,j}^{(r_{i-1,j})} - \frac{\partial y_{i-1,j}^{(r_{i-1,j}+s_{i-1,j})}}{\partial u^{(s_{i-1,j})}} u
\end{aligned} \tag{2.27}$$

for some $j \in \{1, \cdots, 2^{i-1}\}$.

- *If* $\mathcal{D}_{s+2}^* \cap \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_{i1}^{(\ell)}, \ell \geq 0\} = 0$, *then stop!*
- *Let* $\{\mathrm{d}y, \ldots, \mathrm{d}y^{(r-1)}; \ldots; \mathrm{d}y_{i1}, \ldots, \mathrm{d}y_{i1}^{(r_{i1}-1)}\}$ *be a basis for*

$$\mathcal{X}_{i+1,1} := \mathcal{X}_i + \mathcal{D}_{s+2}^* \cap \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_{i1}^{(\ell)}, \ell \geq 0\}$$

where $r_{i1} = \dim\mathcal{X}_{i+1,1} - \dim\mathcal{X}_i$.

- *If* $\mathcal{D}_{s+2}^* \cap \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_{ij}^{(\ell)}, \ell \geq 0\} = 0$ *for* $j = 2, \ldots, 2^{i-1}$, *then stop!*
- *For* $j = 2, \ldots, 2^{i-1}$, *let*

$$\{\mathrm{d}y, \ldots, \mathrm{d}y^{(r-1)}; \ldots; \mathrm{d}y_{ij}, \ldots, \mathrm{d}y_{ij}^{(r_{ij}-1)}\}$$

be a basis for

$$\mathcal{X}_{i+1,j} := \mathcal{X}_{i+1,j-1} + \mathcal{D}^*_{s+2} \cap \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_{ij}^{(\ell)}, \ell \geq 0\}$$

where $r_{ij} = \mathrm{dim}\mathcal{X}_{i+1,j} - \mathrm{dim}\mathcal{X}_{i+1,j-1}$. *Set* $\mathcal{X}_{i+1} = \sum \mathcal{X}_{i+1,j}$

- *If* $\forall \ell \geq r_{ij}, \mathrm{d}y_{ij}^{(r_{ij})} \in \mathcal{X}_{i+1}$, *set* $s_{ij} = -1$.

If $\exists \ell \geq r_{ij}, \mathrm{d}y_{ij}^{(\ell)} \notin \mathcal{X}_{i+1}$, *then define* s_{ij} *as the smallest integer such that, abusing the notation, one has locally*

$$y_{ij}^{(r_{ij}+s_{ij})} = y_{ij}^{(r_{ij}+s_{ij})}(y^{(\lambda)}, y_{ij}^{(\sigma)}, u, \ldots, u^{(s_{ij})})$$

where $0 < \lambda < r$, $0 < \sigma < r_{ij} + s_{ij}$.

- *If* $s_{ij} \geq 0$ *and* $\partial^2 y_{ij}^{(r_{ij}+s_{ij})} / \partial {u^{(s_{ij})}}^2 \neq 0$ *for some* $j = 1, \ldots, 2^{i-1}$, *stop!*
- *If* $\mathcal{X}_{i+1} + \mathcal{U} = \mathcal{Y} + \mathcal{U}$ *and* $\partial^2 y_{ij}^{(r_{ij}+s_{ij})} / \partial {u^{(s_{ij})}}^2 = 0$, *whenever* $s_{ij} \geq 0$, *then the algorithm stops and the realization is completed. Otherwise, define the new* auxiliary *outputs, whenever* $\mathrm{d}(\partial y_{ij}^{(r_{ij}+s_{ij})} / \partial u^{(s_{ij})}) \neq 0$, *respectively,* $\mathrm{d}(y_{ij}^{(r_{ij})} - \frac{\partial y_{ij}^{(r_{ij}+s_{ij})}}{\partial u^{(s_{ij})}} u) \neq 0$*:*

$$y_{i+1,2j-1} = \frac{\partial y_{ij}^{(r_{ij}+s_{ij})}}{\partial u^{(s_{ij})}},\; y_{i+1,2j} = y_{ij}^{(r_{ij})} - \frac{\partial y_{ij}^{(r_{ij}+s_{ij})}}{\partial u^{(s_{ij})}} u$$

End of the algorithm.

Algorithm 2.21 yields the definition of the state $(x_1, \ldots, x_k) = (y^{(\lambda)}, y_{ij}^{(\sigma)})$ where $0 < \lambda < r$, $0 < \sigma < r_{ij} + s_{ij}$. General necessary and sufficient conditions for the existence of an affine state representation are derived from the algorithm as well.

Theorem 2.22. *System (2.14) admits an affine realization if and only if Algorithm 2.21 can be completed, or equivalently,*

- $k > s$ *and*
$$\frac{\partial^2 y^{(k)}}{\partial (u^{(s)})^2} = 0 \tag{2.28}$$
- *for* $s_{ij} \geq 0$ *and any* $r_{ij} > 0$, $i = 1, 2, \ldots, N$, $j = 1, \ldots, 2^i$,
$$\frac{\partial^2 y_{ij}^{(r_{ij}+s_{ij})}}{\partial (u^{(s_{ij})})^2} = 0 \tag{2.29}$$
where y_{ij}, r_{ij}, *and* s_{ij} *are as defined in Algorithm 2.21,*
- *there exists a finite integer* $N \geq 1$ *such that*
$$\sum_{i=1}^{N} \mathcal{X}_i + \mathcal{U} = \mathcal{Y} + \mathcal{U} \tag{2.30}$$

Remark 2.23. Condition (2.29) is mentioned in [28, 158]. It embodies the fact that the input-output equation (2.14) as well as the differential equations relating the auxiliary outputs are affine in the highest time derivative of the input.

Proof (Proof of Theorem 2.22).
Sufficiency: Algorithm 2.21 can be performed if conditions (2.28) and (2.29) are satisfied. State variables are defined in the procedure of the algorithm. This algorithm will be completed in finite steps according to condition (2.30). Consequently, an affine, observable generating system is obtained for the input-output system (2.14).
Necessity: To prove the necessity condition we need a lemma, which is partly contained in [28, 29, 158].

Lemma 2.24. *If there exists a state-space system*

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{2.31}$$

which is a generating system for (2.14), locally around any point $(y_0, ..., u_0^{(s)})$ *in some suitable open dense subset of* $I\!R^{k+s+1}$*, then* $\partial^2 y^{(k)}/\partial {u^{(s)}}^2 = 0$*,* $\mathrm{d}y_{11} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$*, and* $\mathrm{d}y_{12} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$*.*

Proof. It is already known that $\partial^2 y^{(k)}/\partial {u^{(s)}}^2 = 0$ is a necessary condition for the existence of an affine realization of a given input-output system [28, 29, 158].

The rest of the statement follows from the equality

$$\begin{aligned} y^{(k-s)} &= L_f^{k-s} h + [L_g L_f^{k-s-1} h] u \\ &= y_{12} + y_{11} u \end{aligned}$$

∎

If there exists an affine realization, then it can be transformed into the canonical structure displayed by Algorithm 2.21. By Lemma 2.24, (2.28) holds. Condition (2.29) follows from the proof of Lemma 2.24 which is applied to each auxiliary output y_{ij}, considering all state variables in $\mathcal{X}_{i-1}$ as parameters. The realization is observable and the dimension of the state-space is finite, which imply (2.30). ∎

2.8.2 Examples

Example 2.25.

Given the input-output differential equation

$$\ddot{y} = u^2 \sin y \cos y + \dot{u} \sin y \tag{2.32}$$

for which $k = 2$ and $s = 1$, define

$$x_1 = y^{(k-s-1)} = y$$

Let

$$y_{11} = \sin y, \qquad y_{12} = \dot{y} - u \sin y$$

Then $k_{21} = 0$, $k_{22} = 1$. The relation

$$\dot{y}_{12} = -y_{12} u \cos y$$

implies that $s_{22} = 0$. Define

$$x_2 = y_{12}^{(k_{22}-s_{22}-1)} = y_{12}$$

Then a realization of (2.32) is obtained:

$$\begin{cases} \dot{x}_1 = x_2 + (\sin x_1)\, u \\ \dot{x}_2 = -x_2(\cos x_1)\, u \\ y = x_1 \end{cases} \tag{2.33}$$

which is both observable and accessible and therefore it is minimal.

Example 2.26.

Consider the input-output system:

$$u\ddot{y} - u\dot{y}(u^2 - \dot{y}^2)^{1/2} - \dot{y}\dot{u} = 0 \tag{2.34}$$

and write it as

$$\ddot{y} = \dot{y}(u^2 - \dot{y}^2)^{1/2} + \frac{\dot{y}}{u}\dot{u} \tag{2.35}$$

The right-hand side of (2.35) is meromorphic on the open and dense subset of $I\!R^3$, containing the points $(\dot{y}, u, \dot{u})$ such that $u^2 > \dot{y}^2$. Use Algorithm 2.21 to define

$$x_1 = y^{(k-s-1)} = y$$

and define the auxiliary outputs:

$$y_{11} = \frac{\dot{y}}{u}, \qquad y_{12} = \dot{y} - \frac{\dot{y}}{u}u = 0$$

Then,

$$\dot{y}_{11} = y_{11}(1 - y_{11}^2)^{1/2} u$$

Define

$$x_2 = y_{11}$$

A realization is obtained which has the representation:

$$\begin{cases} \dot{x}_1 = x_2 u \\ \dot{x}_2 = x_2(1 - x_2^2)^{1/2} u \\ y = x_1 \end{cases} \tag{2.36}$$

It does not satisfy the strong accessibility rank condition, so it is not a minimal realization.

Example 2.27. [27]

The input-output system

$$\begin{aligned} y^2y^{(3)}u^2 - y^3u^4 - (3\dot{y}/y + 2\dot{u}/u)\ddot{y}y^2u^2 + 2\dot{y}^3u^2 \\ +2\dot{y}\dot{u}^2y^2 + 2\dot{y}^2\dot{u}yu - \dot{y}\ddot{u}y^2u = 0 \end{aligned} \tag{2.37}$$

can be written as

$$\begin{aligned} y^{(3)} = {} & yu^2 + (3\dot{y}/y + 2\dot{u}/u)\ddot{y} - 2\dot{y}^3/y^2 - 2\dot{y}\dot{u}^2/u^2 \\ & -2\dot{y}^2\dot{u}/(yu) + \dot{y}\ddot{u}/u \end{aligned} \tag{2.38}$$

and has been considered before (see Example 2 of [27]). From Step 1 of Algorithm 2.21, $k = 3$, $s = 2$. Let $x_1 = y$ and define $y_{11} = \dot{y}/u$. Then in Step 2 of the algorithm,

$$\ddot{y}_{11} = yu + 3y_{11}\dot{y}_{11}u/y - 2y_{11}^3u^2/y^2 + y_{11}^2\dot{u}/y.$$

So, $k_{11} = 2$ and $s_{11} = 1$. Let $x_2 = y_{11}$ and define

$$y_{22} = \dot{y}_{11} - y_{11}^2u/y$$

Then

$$\dot{y}_{22} = (y + y_{11}y_{22}/y)u$$

Thus $x_1 = y$, $x_2 = y_{11}$, and $x_3 = y_{22}$ yield

$$\begin{cases} \dot{x}_1 = x_2u \\ \dot{x}_2 = x_3 + (x_2^2/x_1)u \\ \dot{x}_3 = (x_1 + x_2x_3/x_1)u \\ \;\, y = x_1 \end{cases} \tag{2.39}$$

Realization (2.39) is different from the realization given in [27] which is not required to fit within the canonical scheme of Algorithm 2.21.

2.9 The Hopping Robot

Consider a hopping robot consisting of a body and a single leg, as sketched in Figure 2.1 . The orientation of the body with respect to the leg is actuated through torque u_1. The length of the leg may vary with the translation of a piston and it is controlled through a force u_2. Although the realization theory was developed for single input systems, it can easily be used to consider this two-input system. It is modeled as follows. Let m be the mass of the leg, J the inertia momentum of the body, r the (variable) length of the leg, θ denotes the angular position of the body, and ϕ the angular position of the leg.

If the action of gravity is neglected, then the mechanical equations yield

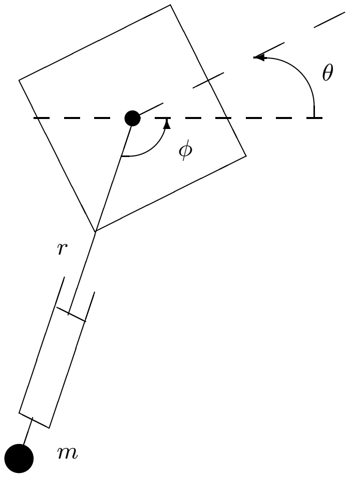

Fig. 2.1. Hopping robot

$$
\begin{aligned}
m\ddot{r} - mr\dot{\phi}^2 &= u_2 \\
J\ddot{\theta} &= u_1 \\
mr^2\ddot{\phi} + 2mr\dot{r}\dot{\phi} &= -u_1
\end{aligned}
\tag{2.40}
$$

Equations (2.40) are higher order input/ouput equations, considering the three outputs (r, θ, ϕ). Construct the extended system Σ_e defined in (2.15).

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \dot{r} \\ r\dot{\phi}^2 \\ \dot{\theta} \\ 0 \\ \dot{\phi} \\ -2\frac{\dot{r}\dot{\phi}}{r} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1/m \\ 0 & 0 \\ 1/J & 0 \\ 0 & 0 \\ -\frac{1}{mr^2} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\tag{2.41}
$$

The latter is not accessible and $\mathcal{H}_\infty$ is spanned by $(2mr\dot{\phi}\mathrm{d}r + mr^2\mathrm{d}\dot{\phi} + J\mathrm{d}\dot{\theta})$. this one-form is exact and equals $\mathrm{d}(mr^2\dot{\phi} + J\dot{\theta})$. This is the kinetic momentum of the hopping robot and is constant. Its minimal realization has not dimension 6. A reduced input-output representation is obtained by

$$
\begin{aligned}
m\ddot{r} - mr\dot{\phi}^2 &= u_2 \\
mr^2\dot{\phi} + J\dot{\theta} &= 0 \\
mr^2\ddot{\phi} + 2mr\dot{r}\dot{\phi} &= -u_1
\end{aligned}
\tag{2.42}
$$

Apply the procedure again, compute the new extended system Σ_e, whose dimension is 5 now, and check $\mathcal{H}_\infty = /$. A minimal realization of the hopping robot (without gravity) thus has dimension 5. Suitable state variables may be chosen as $(r, \dot{r}, \theta, \phi, \dot{\phi})$.

2.10 Some Models

2.10.1 Electromechanical Systems

Consider an inverted pendulum of length l with a point mass m attached at the end of the beam, which is actuated by the torque u applied at the base of the beam. Let g denote the gravitational constant, and φ the angular position of the pendulum with respect to the vertical position. Then the equation of motion is

$$ml^2\ddot{\varphi} - mgl\sin\varphi = u.$$

The angle φ is the output. Rewriting it into the form (2.14),

$$\ddot{\varphi} = \frac{1}{ml^2}[u + mgl\sin\varphi].$$

Algorithm 2.21 can be applied that yields the obvious state variables $x_1 = \varphi$ and $x_2 = \dot{\varphi}$. The state realization is then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \tfrac{1}{ml^2}[u + mgl\sin x_1] \\ \varphi &= x_1 \end{aligned}$$

2.10.2 Virus Dynamics

Several models of virus dynamics can be found in [129]. Let us consider here the HIV infection and the elementary modeling of the immune system when it is subject to HIV infection. The immune system is based on two main actors, the so-called CD8 cells and the CD4 cells. The CD4 cells act as markers, they mark and identify the undesirable agents as viruses, bacteria, etc. The CD8 cells act as killers. However the CD8 cells kill only agents that have been marked beforehand by some CD4 cell. The body is subject to many infectious agents, and the majority of those infections have no consequence at all. Some of them are agressive against specific tissues of the body and the immune system is able to eliminate the infection. What is unfortunate about HIV is that this virus attacks the basis of the immune system itself. The HIV virus infects CD4 cells which will no longer be able to mark the HIV virions. After the population of healthy CD4 cells decreases, the HIV virions will thus be protected against the immune system. Infected CD4 cells act as host cells and they produce new HIV virions. An elementary model may be derived. Let T denote the population of healthy CD4 cells. Let T^* denote the population of infected CD4 cells. Let v denote the population of HIV virions. As any living specie, the CD4 cells have some finite lifetime $1/\delta$. The evolution of some *independent* population is then approximated by the linear first-order system:

$$\dot{T} = -\delta T$$

The body is assumed to produce new CD4 cells at some constant rate s; thus, the evolution of T in a noninfected body will be described by

$$\dot{T} = s - \delta T$$

and the population T stabilizes at some equilibrium $T_0 = s/\delta$. In an infected body, besides natural death, the population T decreases due to the agression of the virus. Part of the healthy CD4 cells will be converted into infected CD4 cells. It is supposed to be proportional to both the T population and the v population. Finally, the dynamics of T is

$$\dot{T} = s - \delta T - \beta T v$$

The population T^* of infected CD4 cells is also submitted to a natural death, with a lifetime $1/\mu$. The only source of production of new infected CD4 cells has already been described and its rate equals $\beta T v$. Thus, the dynamics of T^* reads as

$$\dot{T}^* = \beta T v - \mu T^*$$

The population v of HIV virions is submitted to a natural death and their lifetime equals $1/c$. The production of new virions is proportional to the population T^* of infected CD4 cells. Let us exclude here the case of new external injection of some virus load. Then the dynamics of v becomes

$$\dot{v} = kT^* - cv$$

Problems

2.1. Consider the following "Ball and Beam" system [166], whose input is the angle α and whose output is the ball position r. The input-output equation of the system is

$$0 = \left(\frac{J}{R^2} + m\right)\ddot{r} + mg\sin\alpha - mr\dot{\alpha}^2$$

where the constant parameters J, R, m, g represent, respectively, the inertia of the ball, its radius, its mass, and the gravitational constant.

1. Write a generalized state space representation of the system, if any.
2. Write a classical state-space realization, if any. Hint: Apply Theorem 2.16.

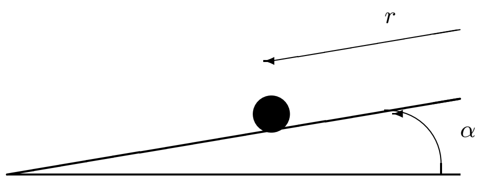

Fig. 2.2. Ball and Beam

2.2. Consider the same "ball and beam" system as in Exercise 2.1 and assume that the angle α is produced by a torque u, so that

$$\ddot{\alpha} = u$$

Considering u as the input and r as the output, write a classical state-space realization.

2.3. Consider the following "Pendulum on a cart" system. Let m and l be the

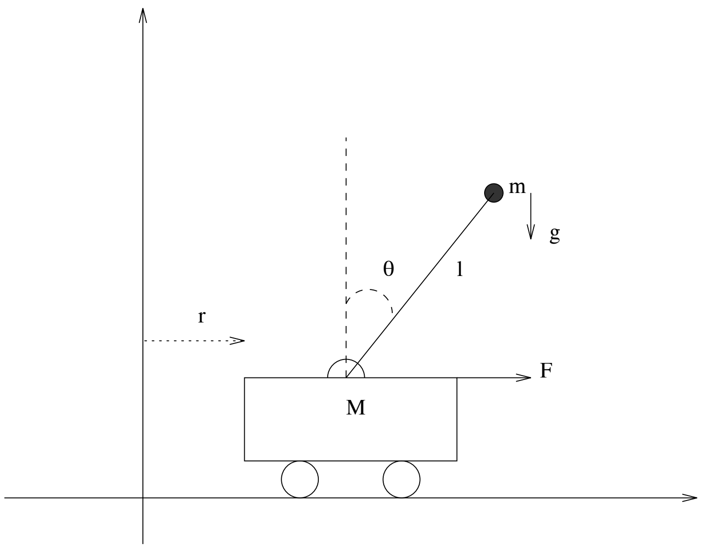

Fig. 2.3. Pendulum on a cart

mass and the length of the pendulum, let M be the mass of the cart. The external force F applied to the cart is the control variable. This system can be modeled as

$$\begin{aligned}(M+m)\ddot{r} + b\dot{r} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta &= F \\ (I+ml^2)\ddot{\theta} + mgl\sin\theta &= -ml\ddot{r}\cos\theta\end{aligned}$$

Considering the output $y = \theta$, write a classical state-space realization, if any.