Fields Institute Monographs 34

The Fields Institute for Research in Mathematical Sciences

Radu Laza
Matthias Schütt
Noriko Yui
Editors

Calabi–Yau Varieties: Arithmetic, Geometry and Physics

Lecture Notes on Concentrated Graduate Courses

Springer

Fields Institute Monographs

VOLUME 34

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FIELDS The Fields Institute for Research in the Mathematical Sciences

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Editors
Radu Laza
Mathematics Department
Stony Brook University
Stony Brook, NY, USA

Matthias Schütt
Institut für Algebraische Geometrie
Leibniz Universität Hannover
Hannover, Germany

Noriko Yui
Department of Mathematics and Statistics
Queen's University
Kingston, ON, Canada

ISSN 1069-5273 ISSN 2194-3079 (electronic)
Fields Institute Monographs
ISBN 978-1-4939-2829-3 ISBN 978-1-4939-2830-9 (eBook)
DOI 10.1007/978-1-4939-2830-9

Library of Congress Control Number: 2015945196

Mathematics Subject Classification (2010): 14J32, 11Gxx, 24Dxx, 14Cxx, 32-xx, 81-xx

Springer New York Heidelberg Dordrecht London

Cover illustration: Drawing of J.C. Fields by Keith Yeomans

Printed on acid-free paper

Springer Science+Business Media LLC New York is part of Springer Science+Business Media (www.springer.com)

Preface

The thematic program *Calabi–Yau Varieties: Arithmetic, Geometry, and Physics* was held at the Fields Institute for Mathematical Sciences from July 1 to December 31, 2013. It was organized by Mark Gross (UC San Diego/Cambridge), Sergei Gukov (Caltech), Radu Laza (Stony Brook), Matthias Schütt (Hannover), Johannes Walcher (McGill), Shing-Tung Yau (Harvard), and Noriko Yui (Kingston/Fields).

This monograph contains introductory material on Calabi–Yau manifolds and is based on lectures which took place during the introductory period for the workshops of the thematic program. These workshops ("Modular Forms Around String Theory," "Enumerative Geometry and Calabi–Yau Varieties," "Physics Around Mirror Symmetry," "Hodge Theory in String Theory") and consequently the lectures here explore various perspectives on Calabi–Yau varieties. Thus, the title "Calabi–Yau Varieties: Arithmetic, Geometry, and Physics" is quite appropriate.

The goal of this volume is to give a friendly introduction to the rapidly developing and vast research areas concerning Calabi–Yau varieties and string theory. Our hope is that anyone who wishes to work on or is interested in subjects in this area will start with this book. More precisely, we would like to tell prospective graduate students that "This is a book you should read if you are interested in getting into the Calabi–Yau worlds: mathematics and string theory."

The articles presented in this volume have been prepared by young researchers (mostly students and postdocs affiliated with the thematic program) with utmost enthusiasm, based on the concentrated graduate courses given by them during the thematic program. The editors wish to express their great appreciation to all of them for preparing their manuscripts for the Fields Monograph Series, which required extra effort presenting not only current developments but also some background material on the topics discussed. All articles in this volume were peer-reviewed. We are deeply grateful to all the referees for their efforts evaluating the articles, in particular in the limited time frame. This volume was edited by R. Laza, M. Schütt, and N. Yui.

The thematic program was financially supported by various organizations. In addition to the Fields Institute, the program received substantial support from the NSF (DMS-1247441, DMS-125481), the PIMS CRG Program *Geometry and*

Physics, and the Perimeter Institute. Additionally, several participants used their individual grants (e.g., NSF, ERC, or NSERC) to cover their travel expenses. We wish to thank all these institutions for their support.

Last but not least, our thanks go to everyone at the Fields Institute for making this thematic program so successful and enjoyable.

Stony Brook, USA — Radu Laza
Hannover, Germany — Matthias Schütt
Kingston, Canada — Noriko Yui

October 2014

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Contributors

Sara Angela Filippini Institut für Mathematik, Universität Zürich, Zürich, Switzerland

Alberto García-Raboso Department of Mathematics, University of Toronto, 40 St. George Street, Toronto, ON M5S 2E4, Canada

Andrew Harder Department of Mathematical and Statistical Sciences, 632 CAB, University of Alberta, Edmonton, AB, Canada

Atsushi Kanazawa Department of Mathematics, Center for Mathematical Sciences and Applications, Harvard University, Cambridge, MA, USA

Tyler L. Kelly Department of Pure Mathematics and Mathematical Statistics, University of Cambrigde, Wilberforce Road, Cambridge CB3 0WB, UK

Matt Kerr Department of Mathematics, Washington University in St. Louis, St. Louis, MO, USA

D. Peter Overholser Department of Mathematics, KU Leuven, Leuven (Heverlee), Belgium

Andrija Peruničić Department of Mathematics and Statistics, Queen's University, Kingston, ON, Canada

Callum Quigley Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada

Steven Rayan Department of Mathematics, University of Toronto, 40 St. George Street, Toronto, ON M5S 2E4, Canada

Simon C.F. Rose Max Planck Institute for Mathematics, Bonn, Germany

Helge Ruddat Mathematisches Institut, Universität Mainz, Mainz, Germany

Alan Thompson Department of Pure Mathematics, University of Waterloo, Waterloo, ON, Canada

Michel van Garrel KIAS, Seoul, Republic of Korea

Ursula Whitcher Department of Mathematics, Hibbard Humanities Hall 508, University of Wisconsin-Eau Claire, Eau Claire, WI, USA

Jie Zhou Perimeter Institute for Theoretical Physics, Waterloo, ON, Canada

Yuecheng Zhu Mathematics Department, University of Texas at Austin, Austin, TX, USA

Part I
K3 Surfaces: Arithmetic, Geometry and Moduli

The Geometry and Moduli of K3 Surfaces

Andrew Harder and Alan Thompson

1 General Results on K3 Surfaces

We begin by recalling the definition of a K3 surface.

Definition 1. A *K3 surface* S is a smooth compact complex surface with trivial canonical bundle $\omega_S \cong \mathscr{O}_S$ and $h^1(S, \mathscr{O}_S) = 0$.

Remark 1. Note that an arbitrary K3 surface S is not necessarily projective, but every K3 surface is Kähler. This was first proved by Siu [65] who, by treating the K3 case, completed the proof of a conjecture of Kodaira [45, Sect. XII.1] stating that every smooth compact complex surface with even first Betti number is Kähler. A direct proof of this conjecture may be found in [7, Thm. IV.3.1].

Unless otherwise stated, throughout these notes S will denote an arbitrary K3 surface. In the remainder of this section we will study the geometry of S, then use this to initiate our study of the moduli space of K3 surfaces. Our main reference for this section will be [7, Chap. VIII].

A. Harder
Department of Mathematical and Statistical Sciences, 632 CAB, University of Alberta, Edmonton, AB T6G 2G1, Canada
e-mail: aharder@ualberta.ca

A. Thompson (✉)
Department of Pure Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, ON N2L 3G1, Canada
e-mail: am6thomp@uwaterloo.ca

R. Laza et al. (eds.), *Calabi-Yau Varieties: Arithmetic, Geometry and Physics*, Fields Institute Monographs 34, DOI 10.1007/978-1-4939-2830-9_1

1.1 Hodge Theory

We begin by studying the Hodge theory of a K3 surface S. The Hodge diamond of S has the form

$$\begin{matrix} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{matrix} \quad = \quad \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}\,.$$

We note that this is largely trivial: the only interesting behaviour happens in the second cohomology group. As we shall see, the structure of this cohomology group determines the isomorphism class of a K3 surface, so can be used to construct a moduli space for K3 surfaces.

The second cohomology group $H^2(S,\mathbb{Z})$ with the cup-product pairing $\langle\cdot,\cdot\rangle$ forms a lattice isometric to the *K3 lattice*

$$\Lambda_{\mathrm{K3}} := H \oplus H \oplus H \oplus (-E_8) \oplus (-E_8),$$

where H is the hyperbolic plane (an even, unimodular, indefinite lattice of rank 2) and E_8 is the even, unimodular, positive definite lattice of rank 8 corresponding to the Dynkin diagram E_8. The lattice Λ_{K3} is a non-degenerate even lattice of rank 22 and signature $(3,19)$ (for the reader unfamiliar with lattice theory, we have included a short appendix containing results and definitions relevant to these notes).

There are two important sublattices of $H^2(S,\mathbb{Z})$ that appear frequently in the study of K3 surfaces. The first is the *Néron-Severi lattice* $\mathrm{NS}(S)$, given by

$$\mathrm{NS}(S) := H^{1,1}(S) \cap H^2(S,\mathbb{Z})$$

(here we identify $H^2(S,\mathbb{Z})$ with its image under the natural embedding $H^2(S,\mathbb{Z}) \hookrightarrow H^2(S,\mathbb{C})$). By the Lefschetz theorem on $(1,1)$-classes [7, Thm. IV.2.13], $\mathrm{NS}(S)$ is isomorphic to the Picard lattice $\mathrm{Pic}(S)$, with isomorphism induced by the first Chern class map.

The second important sublattice of $H^2(S,\mathbb{Z})$ is the *transcendental lattice* $\mathrm{T}(S)$. It is defined to be the smallest sublattice of $H^2(S,\mathbb{Z})$ whose complexification contains a generator σ of $H^{2,0}(S)$. In the case where $\mathrm{NS}(S)$ is nondegenerate (which happens, for instance, when S is projective), then the transcendental lattice is equal to the orthogonal complement of $\mathrm{NS}(S)$ in $H^2(S,\mathbb{Z})$.

The structure of the second cohomology of S is an important object to study, as it determines the isomorphism class of S.

Theorem 1 (Weak Torelli [7, Cor. VIII.11.2]). *Two K3 surfaces S and S' are isomorphic if and only if there is a lattice isometry $H^2(S,\mathbb{Z}) \to H^2(S',\mathbb{Z})$, whose $\mathbb{C}$-linear extension $H^2(S,\mathbb{C}) \to H^2(S',\mathbb{C})$ preserves the Hodge decomposition* (*such an isometry is called a* Hodge isometry).

1.2 The Period Mapping

We can use the weak Torelli theorem to begin constructing a moduli space for K3 surfaces. We start by defining a *marking* on the K3 surface S.

Definition 2. A *marking* on S is a choice of isometry $\phi\colon H^2(S,\mathbb{Z}) \to \Lambda_{\mathrm{K3}}$. We say that (S,ϕ) is a *marked K3 surface*.

Since the canonical bundle of S is trivial, we have $H^{2,0}(S) := H^0(S,\Omega_S^2) = H^0(S,\mathscr{O}_S)$. Let $\sigma \in H^{2,0}(S)$ be any nonzero element. Then σ is a nowhere vanishing 2-form on S. Using the Hodge decomposition, we may treat σ as an element of $H^2(S,\mathbb{C})$. This cohomology group carries a bilinear form $\langle\cdot,\cdot\rangle$, given by the $\mathbb{C}$-linear extension of the cup-product pairing, with respect to which we have $\langle\sigma,\sigma\rangle = 0$ and $\langle\sigma,\overline{\sigma}\rangle > 0$.

If ϕ is a marking for S and $\phi_{\mathbb{C}}\colon H^2(S,\mathbb{C}) \to \Lambda_{\mathrm{K3}} \otimes \mathbb{C}$ is its $\mathbb{C}$-linear extension, then $\phi_{\mathbb{C}}(H^{2,0}(S))$ is a line through the origin in $\Lambda_{\mathrm{K3}} \otimes \mathbb{C}$ spanned by $\phi_{\mathbb{C}}(\sigma)$. Projectivising, we see that $\phi_{\mathbb{C}}(H^{2,0}(S))$ defines a point in

$$\Omega_{\mathrm{K3}} := \{[\sigma] \in \mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C}) \mid \langle\sigma,\sigma\rangle = 0, \langle\sigma, \overline{\sigma}\rangle > 0\}.$$

Ω_{K3} is a 20-dimensional complex manifold called the *period space of K3 surfaces*. The point defined by $\phi_{\mathbb{C}}(H^{2,0}(S))$ is the *period point of the marked K3 surface* (S,ϕ).

The Weak Torelli theorem (Theorem 1) gives that two K3 surfaces are isomorphic if and only if there are markings for them such that the corresponding period points are the same.

Now we extend this idea to families. Let $\pi\colon \mathscr{S} \to U$ be a flat family of K3 surfaces over a small contractible open set U and let S be a fibre of π. A choice of marking $\phi\colon H^2(S,\mathbb{Z}) \to \Lambda_{\mathrm{K3}}$ for S can be extended uniquely to a marking $\phi_U\colon R^2\pi_*\mathbb{Z} \to (\Lambda_{\mathrm{K3}})_U$ for the family $\mathscr{S}$, where $(\Lambda_{\mathrm{K3}})_U$ denotes the constant sheaf with fibre Λ_{K3} on U. Applying the above construction to the marked K3 surfaces in the family $\mathscr{S}$, we obtain a holomorphic map $U \to \Omega_{\mathrm{K3}}$, called the *period mapping* associated to the family $\pi\colon \mathscr{S} \to U$.

Applying this to the case where $\pi\colon \mathscr{S} \to U$ is a representative of the versal deformation of S, one finds:

Theorem 2 (Local Torelli [7, Thm. VIII.7.3]). *For any marked K3 surface S, the period mapping from the versal deformation space of S to Ω_{K3} is a local isomorphism.*

This shows that the period mapping is well-behaved under small deformations of a marked K3 surface. Moreover, we have:

Theorem 3 (Surjectivity of the Period Map [7, Cor. VIII.14.2]). *Every point of Ω_{K3} occurs as the period point of some marked K3 surface.*