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Javad Mashreghi

Derivatives of Inner Functions

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Ouvrez une école, vous fermerez une prison

Victor Hugo

It was in 1934 that Pahlavi High School was established in my hometown, Kashan. In 1946, the school moved to a new building constructed over a rather vast area and featuring an awesome architectural design. The school was renamed Imam Khomeini High School after the 1979 revolution. Over the years, numerous bright minds were trained in the stimulating environment of this school. Before long, the impressive school building had become a reminder of all the great intellectuals who had either studied or taught there. To the utter regret of the latter, however, the building was completely demolished in 1995, only to give way to the current, incomplete one. I dedicate this monograph to all the caring and respected men, teachers and employees alike, who kept the flame of education alight for many years in this institute.

مدرسه‌ای باز کنید، زندانی را خواهید بست.

ویکتور هوگو

دبیرستان پهلوی در شهر من کاشان به سال ۱۳۱۴ تاسیس و در سال ۱۳۲۶ در مکانی نسبتاً وسیع و با معماری بسیار زیبا و خاطره‌انگیز بنا شد. بعد از انقلاب ۱۳۵۷، نام اولیه دبیرستان به امام خمینی تغییر یافت. این مجموعه در طول سالیان متمادی نخبگان زیادی را در دامن پر مهر خود پرورش داد و بنای با شکوهش یاد و خاطره‌ی مردان بزرگ و گران‌قدری را که در آن تدریس و یا تحصیل نموده بودند زنده نگاه می‌داشت. با کمال تأسف در سال ۱۳۷۴، این بنا به طور کامل تخریب گردید و بنای جدید و ناتمام فعلی به جای آن ساخته شد. این کتاب را به دبیران دلسوز و کارمندان شریفی که مشعل آموزش را در این مرکز برافروخته داشته و به دوش کشیدند تقدیم می‌دارم.

Painting by Mrs. Najmeh Firoozpour

Preface

Infinite Blaschke products were introduced by W. Blaschke in 1915 [9]. In 1929, R. Nevanlinna introduced the class of bounded analytic functions with almost everywhere unimodular boundary values [35]. However, the term *inner function* was coined much later by A. Beurling in his seminal work on the invariant subspaces of the shift operator [8]. The first extensive studies of the properties of inner functions were made by O. Frostman [22], W. Seidel [43] and F. Riesz [40]. Their efforts to understand the zeros and boundary behavior of bounded analytic functions led to the celebrated canonical factorization theorem. The special factorization that we need says that each inner function is the product of a *Blaschke product* and a zero free inner function, the so called *singular inner function*, which is generated by a singular measure residing on the unit circle. Roughly speaking, we can say that the Blaschke product is formed with the zeros of an inner function inside the open unit disc, and the singular part stems from its zeros on the boundary.

In July 2011, E. Fricain and I organized a conference on *Blaschke products and their applications* in the Fields Institute (Toronto). There were several interesting talks about the boundary behavior of inner functions, in particular Blaschke products, and their derivatives. I felt the need to gather some classical results in a short monograph for graduate students and as a handy reference for experts. However, the literature is very vast and it is a difficult task to choose among various important results. For example, the book of P. Colwell [16] can provide a panoramic picture of this subject. Hence, I restricted myself just to the integral means of the derivatives and, even for this narrow subject, I was very selective.

The Fields Institute exclusively supported our conference on Blaschke products, and its direction constantly helped us for the production of the proceedings and this monograph. In particular, I owe profound thanks to Carl Riehm, the Managing Editor of Publications, for his care, guidance, and enthusiastic support. Last but not the least, I would like to deeply thank

Joseph Cima (University of North Carolina), Ian Graham (University of Toronto), and Armen Edigarian (Jagiellonian University) who kindly read the manuscript and made many valuable suggestions. Their remarks enormously improved the quality of text.

Montreal, QC Javad Mashreghi

Contents

Chapter 1
Inner Functions

The theory of Hardy spaces is a well established part of analytic function theory. Inner functions constitute a special family in this category. Therefore, it is natural to start with several topics on Hardy spaces and apply them in our discussions. However, we are not in a position to study this theory in detail and we assume that our readers have an elementary familiarity with this subject. In this chapter, we briefly mention, mostly without proof, the main theorems that we need in the study of inner functions. For a detailed study of this topic, we refer to [33].

1.1 The Poisson Integral of a Measure

Let μ be a complex Borel measure on the unit circle $\mathbb{T}$. Then the Poisson integral of μ on the open unit disc $\mathbb{D}$ is defined by the formula

$$P_\mu(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|z-\zeta|^2}\, d\mu(\zeta), \qquad (z \in \mathbb{D}).$$

If $d\mu(e^{i\theta}) = u(e^{i\theta})\, d\theta/2\pi$, where $u \in L^1(\mathbb{T})$, instead of P_μ we write P_u. It is easy to verify that $h = P_\mu$ is a harmonic function on $\mathbb{D}$. Moreover, using Fubini's theorem and the identity

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|z-e^{i\theta}|^2}\, d\theta = 1, \qquad (z \in \mathbb{D}), \tag{1.1}$$

we see that

$$\frac{1}{2\pi}\int_0^{2\pi} |h(re^{i\theta})|\, d\theta \le \int_{\mathbb{T}} \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{|re^{i\theta}-\zeta|^2}\, d\theta \right) d|\mu|(\zeta) = \|\mu\|,$$

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where $\|\mu\|$ is the total variation of the measure μ on $\mathbb{T}$. Hence, h fulfills the growth restriction

$$\sup_{0\le r<1}\int_0^{2\pi} |h(re^{i\theta})|\,d\theta < \infty. \tag{1.2}$$

Hence, the Poisson integral of a Borel measure on $\mathbb{T}$ is a harmonic function on $\mathbb{D}$ which satisfies (1.2). As a matter of fact, the converse to this assertion is also true and we have the following complete characterization.

Theorem 1.1 (Plessner [36]) *Let h be a function defined on $\mathbb{D}$. Then the following assertions are equivalent:*

(i) h is a harmonic function on $\mathbb{D}$ which satisfies the condition (1.2);
(ii) there exists a (unique) Borel measure μ on $\mathbb{T}$ such that $h = P_\mu$.

As a special case, if μ is positive, then $h = P_\mu$ is a positive harmonic function on $\mathbb{D}$ which satisfies (1.2). And if h is a given positive harmonic function, then, by the mean value property, it satisfies

$$\int_0^{2\pi} |h(re^{i\theta})|\,d\theta = \int_0^{2\pi} h(re^{i\theta})\,d\theta = 2\pi h(0), \qquad (0\le r<1).$$

Therefore, in this case, Theorem 1.1 is rewritten as follows.

Corollary 1.2 (Herglotz [28]) *Let h be a function defined on $\mathbb{D}$. Then the following assertions are equivalent:*

(i) h is a positive harmonic function on $\mathbb{D}$;
(ii) there exists a (unique) positive Borel measure μ on $\mathbb{T}$ such that $h = P_\mu$.

The following celebrated result of Fatou provides a sufficient condition for the existence of radial limits of P_μ.

Theorem 1.3 (Fatou [20]) *Let μ be a complex Borel measure on $\mathbb{T}$. Suppose that at $e^{i\theta}\in\mathbb{T}$ the symmetric derivative*

$$\mu'(e^{i\theta}) = \lim_{t\to 0}\frac{\mu\big(\,\{e^{is} : s\in(\theta-t,\theta+t)\}\,\big)}{2t}$$

exists. Then

$$\lim_{r\to 1} P_\mu(re^{i\theta}) = 2\pi\,\mu'(e^{i\theta}).$$

Proof. Without loss of generality, assume that $\theta = 0$. Put

$$\mathfrak{U}(x) = \mu\big(\,\{e^{is} : s\in[-\pi,x)\}\,\big), \qquad x\in[-\pi,\pi).$$

Then integration by parts gives

$$\begin{aligned}
P_\mu(r) &= \int_{\mathbb{T}} \frac{1-r^2}{1+r^2-2r\cos t}\, d\mu(e^{it}) \\
&= \left\{ \frac{1-r^2}{1+r^2-2r\cos t}\,\mathfrak{U}(t) \right\}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\partial}{\partial t} \left\{ \frac{1-r^2}{1+r^2-2r\cos t} \right\} \mathfrak{U}(t)\, dt \\
&= \frac{1-r}{1+r}\,\mathfrak{U}(\pi) + \int_{-\pi}^{\pi} \frac{(1-r^2)\,2r\,\sin t}{(1+r^2-2r\cos t)^2}\,\mathfrak{U}(t)\, dt \\
&= \frac{1-r}{1+r}\,\mathfrak{U}(\pi) + \frac{2r}{1+r} \int_{-\pi}^{\pi} \frac{(1+r)^2\,(1-r)\,t\,\sin t}{(1+r^2-2r\cos t)^2} \times \frac{\mathfrak{U}(t)-\mathfrak{U}(-t)}{2t}\, dt.
\end{aligned}$$

Let

$$\phi(t) = \frac{\mathfrak{U}(t)-\mathfrak{U}(-t)}{2t} - \mu'(1) = \frac{1}{2t}\int_{-t}^{t} d\mu(e^{is}) - \mu'(1), \qquad (-\pi \le t \le \pi),$$

and note that, by assumption,

$$\lim_{t\to 0} \phi(t) = 0. \tag{1.3}$$

Let

$$F_r(t) = \frac{(1+r)^2\,(1-r)\,t\,\sin t}{(1+r^2-2r\cos t)^2}, \qquad (0 \le r < 1,\ -\pi \le t \le \pi).$$

This function satisfies the following properties:

(i) $F_r \ge 0$;

(ii)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_r(t)\, dt = 1;$$

(iii) for each fixed $0 < \delta < \pi$, we have

$$\lim_{r\to 0} \left(\sup_{\delta < |t| \le \pi} F_r(t) \right) = 0.$$

In technical language, F_r is a positive approximate identity on $[-\pi, \pi]$. Using the new notations, we have

$$\lim_{r\to 1} P_\mu(r) = \lim_{r\to 1} \int_{-\pi}^{\pi} F_r(t)\,\big(\,\phi(t)+\mu'(1)\,\big)\, dt = 2\pi\mu'(1) + \lim_{r\to 1} \int_{-\pi}^{\pi} F_r(t)\,\phi(t)\, dt.$$

By (1.3), given $\varepsilon > 0$, there is δ such that $|\phi(t)| < \varepsilon$, whenever $|t| < \delta$. Without loss of generality, assume that $\delta < \pi$. Then we have

$$\left|\int_{-\pi}^{\pi} F_r(t)\,\phi(t)\ dt\right| \le \int_{-\delta}^{\delta} F_r(t)\,|\phi(t)|\ dt + \int_{\delta<|t|\le\pi} F_r(t)\,|\phi(t)|\ dt$$
$$\le \varepsilon \int_{-\delta}^{\delta} F_r(t)\ dt + \left(\max_{-\pi\le t\le\pi} |\phi(t)|\right) \int_{\delta<|t|\le\pi} F_r(t)\ dt$$
$$\le 2\pi\varepsilon + \pi \left(\max_{-\pi\le t\le\pi} |\phi(t)|\right) \left(\sup_{\delta<|t|\le\pi} F_r(t)\right).$$

Therefore, for each $\varepsilon > 0$,

$$\limsup_{r\to1} \left|\int_{-\pi}^{\pi} F_r(t)\,\phi(t)\ dt\right| \le 2\pi\varepsilon.$$

This fact ensures that

$$\lim_{r\to1} P_\mu(r) = 2\pi\mu'(1).$$

By *Lebesgue's decomposition theorem*, for each complex Borel measure μ, there are a function $u \in L^1(\mathbb{T})$ and a complex singular Borel measure σ such that

$$d\mu(e^{i\theta}) = u(e^{i\theta})\,d\theta/2\pi + d\sigma(e^{i\theta}).$$

Moreover, for almost all $e^{i\theta} \in \mathbb{T}$,

$$\mu'(e^{i\theta}) = \lim_{t\to0} \frac{\mu\big(\,\{e^{is} : s \in (\theta - t, \theta + t)\}\,\big)}{2t} = \frac{u(e^{i\theta})}{2\pi}.$$

Hence, we immediately obtain the following two results. First, if $\mu = \sigma$ is a complex singular Borel measure on $\mathbb{T}$, then

$$\lim_{r\to1} P_\sigma(re^{i\theta}) = 0 \tag{1.4}$$

for almost all $e^{i\theta} \in \mathbb{T}$. Second, if $d\mu = u\,d\theta/2\pi$ is absolutely continuous, then

$$\lim_{r\to1} P_u(re^{i\theta}) = u(e^{i\theta}) \tag{1.5}$$

for almost all $e^{i\theta} \in \mathbb{T}$.

The following variant of Fatou's theorem will also be needed. Since F_r is a positive approximate identity, the proof of Theorem 1.3, with slight modification, works in this case too.

Theorem 1.4 *Let μ be a finite positive Borel measure on $\mathbb{T}$, and let $e^{i\theta} \in \mathbb{T}$ be such that*

$$\mu'(e^{i\theta}) = \lim_{t\to0} \frac{\mu\big(\,\{e^{is} : s \in (\theta - t, \theta + t)\}\,\big)}{2t} = \infty.$$