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Nizar Touzi

Optimal Stochastic Control, Stochastic Target Problems, and Backward SDE

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With Chapter 13 by Agnès Tourin

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Finally, I should like to express
all my love to my family:
Christine, our sons Ali and Héni, and our doughter Lilia,
who accompanied me during this visit to Toronto,
all my thanks to them for their patience while I was preparing
these notes,
and all my apologies for my absence even when I was
physically present...

Contents

Chapter 1
Introduction

These notes have been prepared for the graduate course taught at the Fields Institute, Toronto, during the thematic program on quantitative finance which was held from January to June, 2010.

I would like to thank all the participants to these lectures. It was a pleasure for me to share my experience on this subject with the excellent audience that was offered by this special research semester. In particular, their remarks and comments helped to improve parts of this document and to correct some mistakes.

My special thanks go to Bruno Bouchard, Mete Soner, and Agnès Tourin who accepted to act as guest lecturers within this course. These notes have also benefitted from the discussions with them, and some parts are based on my previous work with Bruno and Mete.

These notes have also benefitted from careful reading by Matheus Grasselli, Pierre Henry-Labordère and Tom Salisbury. I greatly appreciate their help and hope there are not many mistakes left.

I would like to express all my thanks to Matheus Grasselli, Tom Hurd, Tom Salisbury, and Sebastian Jaimungal for the warm hospitality at the Fields Institute and their regular attendance to my lectures.

These lectures present the modern approach to stochastic control problems with a special emphasis on the application in financial mathematics. For pedagogical reason, we restrict the scope of the course to the control of diffusion processes, thus ignoring the presence of jumps.

We first review the main tools from stochastic analysis: Brownian motion and the corresponding stochastic integration theory. This already introduces to the first connection with partial differential equations (PDEs). Indeed, by Itô's formula, a linear PDE pops up as the infinitesimal counterpart of the tower property. Conversely, given a nicely behaved smooth solution, the so-called Feynman–Kac formula provides a stochastic representation in terms of a conditional expectation.

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We then introduce the class of standard stochastic control problems where one wishes to maximize the expected value of some gain functional. The first main task is to derive an original weak dynamic programming principle which avoids the heavy measurable selection arguments in typical proofs of the dynamic programming principle when no a priori regularity of the value function is known. The infinitesimal counterpart of the dynamic programming principle is now a nonlinear PDE which is called dynamic programming equation or Hamilton–Jacobi–Bellman equation. The hope is that the dynamic programming equation provides a complete characterization of the problem, once complemented with appropriate boundary conditions. However, this requires strong smoothness conditions, which can be seen to be violated in simple examples.

A parallel picture can be drawn for optimal stopping problems and, in fact, for the more general control and stopping problems. In these notes we do not treat such mixed control problems, and we rather analyze separately these two classes of control problems. Here again, we derive the dynamic programming principle and the corresponding dynamic programming equation under strong smoothness conditions. In the present case, the dynamic programming equation takes the form of the obstacle problem in PDEs.

When the dynamic programming equation happens to have an explicit smooth solution, the verification argument allows to verify whether this candidate indeed coincides with the value function of the control problem. The verification argument provides as a by-product an access to the optimal control, i.e., the solution of the problem. But of course, such lucky cases are rare, and one should not count on solving any stochastic control problem by verification.

In the absence of any general a priori regularity of the value function, the next development of the theory is based on viscosity solutions. This beautiful notion was introduced by Crandal and Lions and provides a weak notion of solutions to second-order degenerate elliptic PDEs. We review the main tools from viscosity solutions which are needed in stochastic control. In particular, we provide a difficulty-incremental presentation of the comparison result (i.e., maximum principle) which implies uniqueness.

We next show that the weak dynamic programming equation implies that the value function is a viscosity solution of the corresponding dynamic programming equation in a wide generality. In particular, we do not assume that the controls are bounded. We emphasize that in the present setting, there is no a priori regularity of the value function needed to derive the dynamic programming equation: we only need it to be locally bounded! Given the general uniqueness results, viscosity solutions provide a powerful tool for the characterization of stochastic control and optimal stopping problems.

The remaining part of the lectures focus on the more recent literature on stochastic control, namely stochastic target problems. These problems are motivated by the superhedging problem in financial mathematics. Various extensions have been studied in the literature. We focus on a particular setting where the proofs are simplified while highlighting the main ideas.

The use of viscosity solutions is crucial for the treatment of stochastic target problems. Indeed, deriving any a priori regularity seems to be a very difficult task. Moreover, by writing formally the corresponding dynamic programming equation and guessing an explicit solution (in some lucky case), there is no known direct verification argument as in standard stochastic control problems. Our approach is then based on a dynamic programming principle suited to this class of problems, and called geometric dynamic programming principle, due to a further extension of stochastic target problems to front propagation problems in differential geometry. The geometric programming principle allows to obtain a dynamic programming equation in the sense of viscosity solutions. We provide some examples where the analysis of the dynamic programming equation leads to a complete solution of the problem.

We also present an interesting extension to stochastic target problems with controlled probability of success. A remarkable trick allows to reduce these problems to standard stochastic target problems. By using this methodology, we show how one can solve explicitly the problem of quantile hedging which was previously solved by Föllmer and Leukert [21] by duality methods in the standard linear case in financial mathematics.

A further extension of stochastic target problems consists in involving the quadratic variation of the control process in the controlled state dynamics. These problems are motivated by examples from financial mathematics related to market illiquidity and are called second-order stochastic target problems. We follow the same line of arguments by formulating a suitable geometric dynamic programming principle and deriving the corresponding dynamic programming equation in the sense of viscosity solutions. The main new difficulty here is to deal with the short-time asymptotics of double stochastic integrals.

The final part of the lectures explores a special type of stochastic target problems in the non-Markovian framework. This leads to the theory of backward stochastic differential equations (BSDE) which was introduced by Pardoux and Peng [32]. Here, in contrast to stochastic target problems, we insist on the existence of a solution to the stochastic target problem. We provide the main existence, uniqueness, stability, and comparison results. We also establish the connection with stochastic control problems. We finally show the connection with semilinear PDEs in the Markovian case.

The extension of the theory of BSDEs to the case where the generator is quadratic in the control variable is very important in view of the applications to portfolio optimization problems. However, the existence and uniqueness can not be addressed as simply as in the Lipschitz case. The first existence and uniqueness results were established by Kobylanski [26] by adapting to the non-Markovian framework techniques developed in the PDE literature. Instead of this highly technical argument, we report the beautiful argument recently developed by Tevzadze [39] and provide applications in financial mathematics.

The final chapter is dedicated to numerical methods for nonlinear PDEs. We provide a complete proof of convergence based on the Barles–Souganidis motone scheme method. The latter is a beautiful and simple argument which exploits the stability of viscosity solutions. Stronger results are provided in the semilinear case by using techniques from BSDEs.