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Homogenization of Partial Differential Equations

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Preface

This book is devoted to homogenization problems for partial differential equations describing various physical phenomena in microinhomogeneous media. This direction in the theory of partial differential equations has been intensively developed for the last forty years; it finds numerous applications in radiophysics, filtration theory, rheology, elasticity theory, and many other areas of physics, mechanics, and engineering sciences.

A medium is called *microinhomogeneous* if its local parameters can be described by functions rapidly varying with respect to the space variables. We will always assume that the length scale of oscillations is much less than the linear sizes of the domain in which a physical process is considered but much greater than the sizes of molecules, so that the process can be described using the differential equations of the mechanics of solids. These differential equations either have rapidly oscillating coefficients (with respect to the space variables) or are considered in domains with complex microstructure, such as *domains with fine-grained boundary* [112] (called later by the better-known term *strongly perforated domains*). The microstructure is understood as the local structure of a domain or the coefficients of equations in the scale of microinhomogeneities.

Obviously, it is practically impossible to solve the corresponding boundary (initial boundary) value problems by either analytical or numerical methods. However, if the microscale is much less than the characteristic scale of the process under investigation (e.g., the wavelength), then it is possible to give a macroscopic description of the process. If it is the case, the medium usually has stable characteristics (heat conductivity, dielectric permeability, etc.), which, in general, may differ substantially from the local characteristics. Such stable characteristics are referred to as homogenized, or effective, characteristics, because they are usually determined by methods of the homogenization theory for differential equations or the relevant mean field methods, effective medium methods, etc.

The term *homogenization* is associated, first of all, with methods of nonlinear mechanics and ordinary differential equations developed by Poincaré, Krylov, Bogolyubov, and Mitropolskii (see, e.g., [21, 123]). For partial differential equations, homogenization problems have been studied by physicists from Maxwell's times, but they remained for a long time outside the interests of mathematicians. However, since the mid 1960s, homogenization theory for partial differential equations began to be intensively developed by mathematicians as well, which was motivated not only by numerous applications (first of all, in the theory of composite media [142]) but also by the emergence of new deep ideas and concepts important for mathematics itself. Currently, there is a great number of publications devoted to mathematical aspects of homogenization such as asymptotic analysis, two-scale convergence, G-convergence, and Γ-convergence. Making no claim to cite all of the available monographs on the subject, we would like to mention the books by Allaire [3], Bakhvalov and Panasenko [9], Bensoussan, Lions, and Papanicolaou [13], Braides and Defranceschi [26], Cioranescu and Donato [42], Cioranescu and Saint Jean Paulin [45], Dal-Maso [46], Marchenko and Khruslov [113], Oleinik, Iossifiyan, and Shamaev [131], Pankov [133], Sanches-Palencia [148], Skrypnik [161], Zhikov, Kozlov, and Oleinik [181].

In the mathematical description of a physical phenomenon in microinhomogeneous mediá, the local characteristics depends on a small parameter ε, which is the characteristic scale of the microstructure. It is the asymptotic analysis, as $\varepsilon \to 0$, of the problem that leads to the homogenized model of the process. It turns out that the limits of solutions of the original problem can be described by certain new differential equations with coefficients smoothly varying in simple domains. These equations constitute a mathematical model of the physical process in a microinhomogeneous medium, their coefficients being effective characteristics of the medium. For example, in the simplest case, the local characteristics of a microinhomogeneous medium are described by periodic functions of the form $a\left(\frac{x}{\varepsilon}\right)$, $x \in \mathbb{R}^n$. The corresponding effective characteristics appear to be independent of x; moreover, the homogenized equations have the same structure as the original ones. Therefore, in this case, the main problem of mathematical modeling is to determine the coefficients of the homogenized equations; these coefficients can then be viewed as the effective parameters of the medium. This situation is typical for various microinhomogeneous media encountered in nature.

However, there exist media with more complicated microstructure, the macroscopic description of which cannot be reduced to the determination of the effective characteristics only, since homogenization leads to equations substantially different from the original ones. Such a situation usually occurs when the microstructure is characterized by several small parameters, of different order of smallness; artificial composite materials as well as some natural media provide the relevant examples. The corresponding homogenized models differ substantially from the original, "microscopic," ones; depending on the microstructure, they appear to be either nonlocal models or multicomponent models or models with memory. This book is basically devoted to the study of structure of microinhomogeneous media leading to "nonstandard" models; therefore, it has almost no intersections with the monographs cited above, except [113]. We began to write this book (which was initially thought of as a revised edition of [113]) in the late 1980s; but since then, new results have been obtained, which now constitute the main contents of the book, the needed results from [113] being presented in more convenient fashion.

In the book, we restrict ourselves mainly to physical phenomena described by the Dirichlet and Neumann boundary value problems in strongly perforated domains and by linear elliptic and parabolic differential equations with rapidly oscillating coefficients; but the developed methods can be applied as well in the study of boundary value problems of elasticity theory, electrodynamics, Fourier boundary value problems, nonlinear problems, etc.

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Kharkov,
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Contents

Homogenization of Partial Differential Equations

1

Introduction

In contrast to the majority of available monographs on homogenization theory dealing with media of relatively simple microstructure (such as periodic, or close to periodic, structures depending on a single small parameter), in this book we study phenomena in media of arbitrary microstructure characterized by several small parameters (or even more complicated media). For such media, homogenized models of physical processes may have various forms differing substantially from an original model. In order to give some ideas about the possible types of models and the topology of microstructure of the corresponding media, in this introduction we consider typical examples of microstructures leading, in the limit, to particular homogenized models. To be specific, we consider a nonstationary heat conduction process (a nonstationary diffusion), described by the heat equation, in microinhomogeneous media of various types.

In the main part of the book (Chapters 2–8), we will consider problems in the general setting and present necessary and sufficient conditions for the convergence of solutions of the original problems to solutions of the corresponding homogenized equations. These condition are formulated in terms of local "mean" characteristics of the microstructure ("mesocharacteristics"), which are then used to express the coefficients of the limiting equations. These characteristics are introduced in cubes ("mesocubes"), which are small relative to the whole domain but at the same time are large relative to the microscale. Since we define the mesocharacteristics following the penalty method and therefore they may seem to be introduced somewhat artificially, we present in this introduction a certain motivation for our approach. Here we also discuss, without proofs, the basic notions needed for the characterization of general microstructures such as the notions of *strongly connected domains* and *weakly connected domains*.

1.1 The Simplest Homogenized Model

We begin with the study of a two-phase medium consisting of a bulk homogeneous material in which small grains (inclusions) of another homogeneous material are embedded. More precisely, let Ω and G be bounded domains in the n-dimensional space $\mathbb{R}^n$ ($n \geq 2$) with smooth boundaries $\partial\Omega$ and ∂G, respectively, $\overline{G}$ (the closure of G) lying in the parallelogram $\Pi = \left\{x \in \mathbb{R}^n : |x_i| < \frac{h_i}{2}\right\}$. Construct the disjoint domains

$$G_{j\varepsilon} = \varepsilon G + \varepsilon \sum_{k=1}^{n} h_k m_{jk} e^k, \quad j = 1, 2, \dots, \tag{1.1}$$

arranged periodically in $\mathbb{R}^n$; here m_{jk} are entire numbers, $\left\{e^k\right\}_{k=1}^{n}$ is an orthonormal basis in $\mathbb{R}^n$, and $\varepsilon > 0$ is a small parameter.

Assume that the bulk material occupies the domain $\Omega_{0\varepsilon} = \Omega \setminus \cup_j \overline{G}_{j\varepsilon}$, whereas the inclusions occupy the domain $\Omega_{1\varepsilon} = \cup_j G_{j\varepsilon}$, where the union is taken over a finite number of domains $G_{j\varepsilon}$ lying entirely in Ω, i.e., $\overline{G}_{j\varepsilon} \subset \Omega$ for $j = 1, 2, \dots, N(\varepsilon) < \infty$; see Figure 1.1.

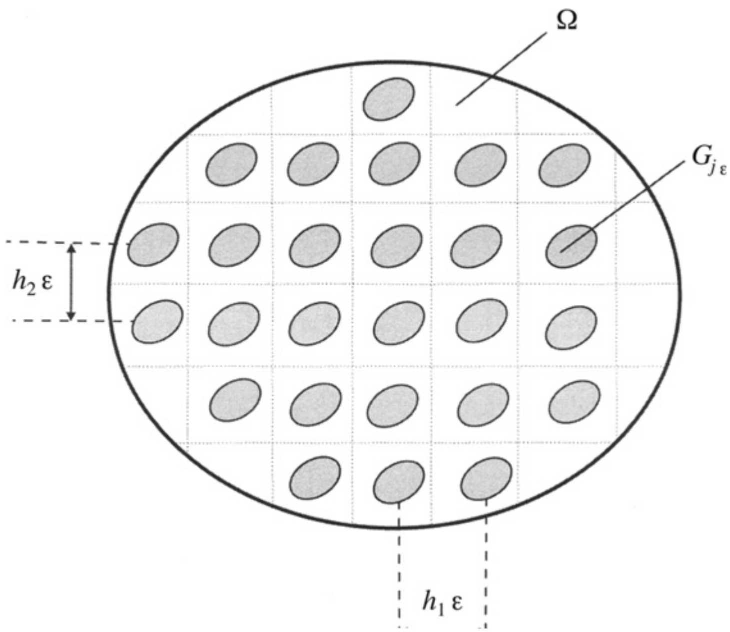

Fig. 1.1.

Denote by a_k and b_k ($k = 0, 1$) the heat conductivity and heat capacity, respectively, of the bulk material ($k = 0$) and the inclusions ($k = 1$), i.e., the local characteristics of the phases.

A nonstationary heat conduction process in such a medium can be described by the temperature function $u^\varepsilon(x, t)$ ($x \in \Omega, t > 0$), which, assuming that there are no