Chapter 2

Borel-fixed monomial ideals

Squarefree monomial ideals occur mostly in combinatorial contexts. The ideals to be studied in this chapter, namely the Borel-fixed monomial ideals, have, in contrast, a more direct connection to algebraic geometry, where they arise as fixed points of a natural algebraic group action on the Hilbert scheme. The fact that we will not treat these schemes until Chapter 18 should not cause any worry—one need not know what the Hilbert scheme is to understand both the group action and its fixed points. After an introductory section concerning group actions on ideals, there are three main themes in this chapter: the construction of generic initial ideals, the minimal resolution of Borel-fixed ideals due to Eliahou–Kervaire, and the Bigatti–Hulett Theorem on extremal behavior of lexicographic segment ideals.

2.1 Group actions

Throughout this chapter, the ground field $\Bbbk$ is assumed to have characteristic 0, and all ideals of the polynomial ring $S = \Bbbk[x_1, \ldots, x_n]$ that we consider are homogeneous with respect to the *standard $\mathbb{Z}$-grading* (an $\mathbb{N}$-grading) given by $\deg(x_i) = 1$ for $i = 1, \ldots, n$. Consider the following inclusion of matrix groups:

$$\begin{array}{ccll} GL_n(\Bbbk) & = & \{\text{invertible } n \times n \text{ matrices}\} & \text{general linear group} \\ \cup & & & \\ B_n(\Bbbk) & = & \{\text{upper triangular matrices}\} & \text{Borel group} \\ \cup & & & \\ T_n(\Bbbk) & = & \{\text{diagonal matrices}\} & \text{algebraic torus group} \end{array}$$

The general linear group (and hence its subgroups) acts on the polynomial ring as follows. For an invertible matrix $g = (g_{ij}) \in GL_n(\Bbbk)$ and a polynomial $f = p(x_1, \ldots, x_n) \in S$, let g act on f by

$$g \cdot p \;=\; p(gx_1, \ldots, gx_n), \quad \text{where} \quad gx_i = \sum_{j=1}^{n} g_{ij} x_j .$$

Given an ideal $I \subset S$, we get a new ideal by applying g to every element of I:

$$g \cdot I \quad = \quad \{ g \cdot p \mid p \in I \}.$$

If I is an ideal with special combinatorial structure and the matrix g is fairly general, then passing from I to $g \cdot I$ will usually lead to a considerable increase in complexity. For a simple example, take $n = 4$ and let I be the principal ideal generated by the quadric $x_1x_2 - x_3x_4$. Then $g \cdot I$ is the principal ideal generated by

$$\begin{aligned}
(g_{11}g_{21} &- g_{31}g_{41})x_1^2 + (g_{12}g_{22} - g_{32}g_{42})x_2^2 \\
&+(g_{13}g_{23} - g_{33}g_{43})x_3^2 + (g_{14}g_{24} - g_{34}g_{44})x_4^2 \\
&+(g_{11}g_{22} - g_{32}g_{41} + g_{12}g_{21} - g_{31}g_{42})x_1x_2 \\
&+(g_{13}g_{21} + g_{11}g_{23} - g_{33}g_{41} - g_{31}g_{43})x_1x_3 \\
&+(g_{14}g_{21} - g_{31}g_{44} - g_{34}g_{41} + g_{11}g_{24})x_1x_4 \\
&+(g_{12}g_{23} - g_{33}g_{42} + g_{13}g_{22} - g_{32}g_{43})x_2x_3 \\
&+(g_{14}g_{22} - g_{34}g_{42} - g_{32}g_{44} + g_{12}g_{24})x_2x_4 \\
&+(g_{13}g_{24} + g_{14}g_{23} - g_{34}g_{43} - g_{33}g_{44})x_3x_4.
\end{aligned}$$

We are interested in ideals I that are fixed under the actions of the three kinds of matrix groups. Let us start with the smallest of these three.

Proposition 2.1 *A nonzero ideal I inside S is fixed under the action of the torus $T_n(\Bbbk)$ if and only if I is a monomial ideal.*

Proof. Torus elements map each variable—and hence each monomial—to a multiple of itself, so monomial ideals are fixed by $T_n(\Bbbk)$. Conversely, let I be an arbitrary torus-fixed ideal, and suppose that $p = \sum c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}}$ is a polynomial in I. Then $\mathbf{t} \cdot p = \sum c_{\mathbf{a}}\mathbf{t}^{\mathbf{a}}\mathbf{x}^{\mathbf{a}}$ is also in I, for every diagonal matrix $\mathbf{t} = \mathrm{diag}(t_1, \ldots, t_n)$. Let $\mathcal{T} = \{\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(s)}\} \subset T_n(\Bbbk)$ be a generic set of diagonal matrices $\mathbf{t}^{(k)} = \mathrm{diag}(t_1^{(k)}, \ldots, t_n^{(k)})$, where the cardinality s equals the number of monomials with nonzero coefficient in p. For each monomial $\mathbf{x}^{\mathbf{a}}$ appearing in p and each diagonal matrix $\mathbf{t} \in \mathcal{T}$, there is a corresponding monomial $\mathbf{t}^{\mathbf{a}}$. Form the $s \times s$ matrix $(\mathbf{t}^{\mathbf{a}})$ whose columns are indexed by the monomials appearing in p and whose rows are indexed by $\mathcal{T}$. As a polynomial in the $n \cdot s$ symbols $\{t_1^{(k)}, \ldots, t_n^{(k)} \mid k = 1, \ldots, s\}$, the determinant of $(\mathbf{t}^{\mathbf{a}})$ is nonzero, because all terms in the expansion are distinct. Hence $\det(\mathbf{t}^{\mathbf{a}}) \neq 0$, because $\mathcal{T}$ is generic. Multiplying the inverse of $(\mathbf{t}^{\mathbf{a}})$ with the column vector whose entries are the polynomials $\mathbf{t} \cdot p$ for $\mathbf{t} \in \mathcal{T}$ yields the column vector whose entries are precisely the terms $c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}}$ appearing in p. We have therefore produced each term $c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}}$ in p as a linear combination of polynomials $\mathbf{t} \cdot p \in I$. It follows that I is a monomial ideal. $\square$

Corollary 2.2 *A nonzero ideal I in S is fixed under the action of the general linear group $GL_n(\Bbbk)$ if and only if I is a power $\mathfrak{m}^d$ of the irrelevant maximal ideal $\mathfrak{m} = \langle x_1, \ldots, x_n\rangle$, for some positive integer d.*

Proof. The vector space of homogeneous polynomials of degree d is fixed by $GL_n(\Bbbk)$, and hence so is the ideal $\mathfrak{m}^d$ it generates. Conversely, suppose I is a $GL_n(\Bbbk)$-fixed ideal and that p is a nonzero polynomial in I of minimal degree, say d. For a general matrix g, the polynomial $g \cdot p$ contains all monomials of degree d in S. Since $g \cdot p$ is in I, and since I is a monomial ideal by Proposition 2.1, every monomial of degree d lies in I. But I contains no nonzero polynomial of degree strictly less than d, so $I = \mathfrak{m}^d$. $\square$

The characterization of monomial ideals in Proposition 2.1 is one of our motivations for having included a chapter on *toric varieties* later in this book: toric varieties are closures of T_n orbits. In representation theory and in the study of determinantal ideals in Part III, one is also often interested in actions of the Borel group B_n. Since B_n contains the torus T_n, and T_n-fixed ideals are monomial, every Borel-fixed ideal is necessarily a monomial ideal. Borel-fixed ideals enjoy the extra property that larger-indexed variables can be swapped for smaller ones without leaving the ideal.

Proposition 2.3 *The following are equivalent for a monomial ideal I.*

(i) I is Borel-fixed.
(ii) If $m \in I$ is any monomial divisible by x_j, then $m\frac{x_i}{x_j} \in I$ for $i < j$.

Proof. Suppose that I is a Borel-fixed ideal. Let $m \in I$ be any monomial divisible by x_j and consider any index $i < j$. Let g be the elementary matrix in $B_n(\Bbbk)$ that sends x_j to $x_j + x_i$ and that fixes all other variables. The polynomial $g \cdot m$ lies in $I = g \cdot I$, and the monomial mx_i/x_j appears in the expansion of $g \cdot m$. Since I is a monomial ideal, this implies that the monomial mx_i/x_j lies in I. We have proved the implication (i) $\Rightarrow$ (ii).

Suppose that condition (ii) holds for a monomial ideal I. Let m be any monomial in I and $g \in B_n(\Bbbk)$ any upper triangular matrix. Every monomial appearing in $g \cdot m$ can be obtained from the monomial m by a sequence of transformations as in (ii). All of these monomials lie in I. Hence $g \cdot m$ lies in I. Therefore condition (i) holds for I. $\square$

In checking whether a given ideal I is Borel-fixed, it suffices to verify condition (ii) for minimal generators m of the ideal I. Hence condition (ii) constitutes an explicit finite algorithm for checking whether I is Borel-fixed.

Example 2.4 Here is a typical Borel-fixed ideal in three variables:

$$I \quad = \quad \langle x_1^2,\, x_1x_2,\, x_2^3,\, x_1x_3^3\rangle.$$

Each of the four generators satisfies condition (ii). The ideal I has the following unique irreducible decomposition (see Chapter 5.2 if these are unfamiliar), which is also a primary decomposition:

$$I = \langle x_1, x_2^3 \rangle \cap \langle x_1^2, x_2, x_3^3 \rangle.$$

The second irreducible component is not Borel-fixed. $\diamond$

The previous example is slightly surprising from the perspective of monomial primary decomposition. Torus-fixed ideals, namely monomial ideals, always admit decompositions as intersections of irreducible torus-fixed ideals; but the same statement does not hold for Borel-fixed ideals.

2.2 Generic initial ideals

This section serves mainly as motivation for studying Borel-fixed ideals, although it is also a convenient place to recall some fundamentals of Gröbner bases, which will be used sporadically throughout the book. The crucial point about Borel-fixed ideals is Theorem 2.9, which says that they arise naturally as initial ideals after generic changes of coordinates. Although this result and the existence of generic initial ideals are stated precisely, we refer the reader elsewhere for large parts of the proof. For a more detailed introduction to Gröbner bases, see [CLO97] or [Eis95, Chapter 15].

To find Gröbner bases, one must first fix a *term order* $<$ on the polynomial ring $S = \Bbbk[x_1, \ldots, x_n]$. By definition, $<$ is a total order on the monomials of S that is *multiplicative*, meaning that $\mathbf{x^b} < \mathbf{x^c}$ if and only if $\mathbf{x^{a+b}} < \mathbf{x^{a+c}}$, and *artinian*, meaning that $1 < \mathbf{x^a}$ for all nonunit monomials $\mathbf{x^a} \in S$. Unless stated otherwise, we assume that our chosen term order satisfies $x_1 > x_2 > \cdots > x_n$.

Given a polynomial $f = \sum_{\mathbf{a} \in \mathbb{N}^n} c_{\mathbf{a}} \mathbf{x^a}$, the monomial $\mathbf{x^a}$ that is largest under the term order $<$ among those whose coefficients are nonzero in p determines the *initial term* $\mathsf{in}_<(f) = c_{\mathbf{a}} \mathbf{x^a}$. When the term order has been fixed for the discussion, we sometimes write simply $\mathsf{in}(f)$. If I is an ideal in S, then the *initial ideal* of I,

$$\mathsf{in}(I) \quad = \quad \langle \mathsf{in}(f) \mid f \in I \rangle,$$

is generated by the set of initial terms of all polynomials in I.

Definition 2.5 Suppose that $I = \langle f_1, \ldots, f_r \rangle$. The set $\{f_1, \ldots, f_r\}$ of generators constitutes a **Gröbner basis** if the initial terms of $f_1, \ldots, f_r$ generate the initial ideal of I; that is, if $\mathsf{in}(I) = \langle \mathsf{in}(f_1), \ldots, \mathsf{in}(f_r) \rangle$.

Every ideal in S has a (finite) Gröbner basis for every term order, because $\mathsf{in}(I)$ is finitely generated by Hilbert's basis theorem. Note that there is no need to mention any ideals when we say, "The set $\{f_1, \ldots, f_r\}$

is a Gröbner basis," as the set must be a Gröbner basis for the ideal $I = \langle f_1, \ldots, f_r \rangle$ it generates. On the other hand, most ideals have many different Gröbner bases for a fixed term order. This uniqueness issue can be resolved by considering a *reduced* Gröbner basis $\{f_1, \ldots, f_r\}$, which means that $\mathsf{in}(f_i)$ has coefficient 1 for each $i = 1, \ldots, r$, and that the only monomial appearing anywhere in $\{f_1, \ldots, f_r\}$ that is divisible by the initial term $\mathsf{in}(f_i)$ is $\mathsf{in}(f_i)$ itself; see Exercise 2.5.

In the proof of the next lemma, we will use a general tool due to Weispfenning [Wei92] for establishing finiteness results in Gröbner basis theory. Suppose that $\mathbf{y}$ is a set of variables different from $x_1, \ldots, x_n$, and let J be an ideal in $S[\mathbf{y}]$, which is the polynomial ring over $\Bbbk$ in the variables $\mathbf{x}$ and $\mathbf{y}$. Every $\Bbbk$-algebra homomorphism $\phi : \Bbbk[\mathbf{y}] \to \Bbbk$ determines a homomorphism $\phi_S : S[\mathbf{y}] \to S$ that sends the $\mathbf{y}$ variables to constants. The image $\phi_S(J)$ is an ideal in S. Given a fixed term order $<$ on S (not on $S[\mathbf{y}]$), Weispfenning proves that J has a *comprehensive Gröbner basis*, meaning a finite set $\mathcal{C}$ of polynomials $p(\mathbf{x}, \mathbf{y}) \in J$ such that for every homomorphism $\phi : \Bbbk[\mathbf{y}] \to \Bbbk$, the specialized set $\phi_S(\mathcal{C})$ is a Gröbner basis for the specialized ideal $\phi_S(J)$ in S with respect to the term order $<$.

Returning to group actions on S, every matrix $g \in GL_n(\Bbbk)$ determines the initial monomial ideal $\mathsf{in}(g \cdot I)$. After fixing a term order, we call two matrices g and g' *equivalent* if

$$\mathsf{in}(g \cdot I) \quad = \quad \mathsf{in}(g' \cdot I).$$

The resulting partition of the group $GL_n(\Bbbk)$ into equivalence classes is a geometrically well-behaved stratification, as we shall now see.

To explain the geometry, we need a little terminology. Let $\mathbf{g} = (g_{ij})$ be an $n \times n$ matrix of indeterminates, so that the algebra $\Bbbk[\mathbf{g}]$ consists of (some of the) polynomial functions on $GL_n(\Bbbk)$. The term *Zariski closed set* inside of $GL_n(\Bbbk)$ or $\Bbbk^n$ refers to the zero set of an ideal in $\Bbbk[\mathbf{g}]$ or S. If V is a Zariski closed set, then a *Zariski open subset* of V refers to the complement of a Zariski closed subset of V.

Lemma 2.6 *For a fixed ideal I and term order $<$, the number of equivalence classes in $GL_n(\Bbbk)$ is finite. One of these classes is a nonempty Zariski open subset U inside of $GL_n(\Bbbk)$.*

Proof. Consider the polynomial ring $S[g_{11}, \ldots, g_{nn}] = \Bbbk[\mathbf{g}, \mathbf{x}]$ in $n^2 + n$ unknowns. Suppose that $p_1(\mathbf{x}), \ldots, p_r(\mathbf{x})$ are generators of the given ideal I in S. Let J be the ideal generated by the elements $\mathbf{g} \cdot p_1(\mathbf{x}), \ldots, \mathbf{g} \cdot p_r(\mathbf{x})$ in $\Bbbk[\mathbf{g}, \mathbf{x}]$, and fix a comprehensive Gröbner basis $\mathcal{C}$ for J.

The equivalence classes in $GL_n(\Bbbk)$ can be read off from the coefficients of the polynomials in $\mathcal{C}$. These coefficients are polynomials in $\Bbbk[\mathbf{g}]$. By requiring that $\det(\mathbf{g}) \neq 0$ and by imposing the conditions "$= 0$" and "$\neq 0$" on these coefficient polynomials in all possible ways, we can read off all possible initial ideals $\mathsf{in}(g \cdot I)$. Since $\mathcal{C}$ is finite, there are only finitely many possibilities, and hence the number of distinct ideals $\mathsf{in}(g \cdot I)$ as g runs over $GL_n(\Bbbk)$ is finite. The unique Zariski open equivalence class U can be specified by imposing the condition "$\neq 0$" on all the leading coefficients of the polynomials in the comprehensive Gröbner basis $\mathcal{C}$. □

The previous lemma tells us that the next definition makes sense.

Definition 2.7 Fix a term order $<$ on S. The initial ideal $\mathsf{in}_<(g \cdot I)$ that, as a function of g, is constant on a Zariski open subset U of GL_n is called the **generic initial ideal** of I for the term order $<$. It is denoted by

$$\mathsf{gin}_<(I) \quad = \quad \mathsf{in}_<(g \cdot I).$$

Example 2.8 Let $n = 2$ and consider the ideal $I = \langle x_1^2, x_2^2 \rangle$, where $<$ is the lexicographic order with $x_1 > x_2$. For this term order, the ideal J defined in the proof of Lemma 2.6 has the comprehensive Gröbner basis

$$\begin{aligned} \mathcal{C} \;=\; \{ & g_{11}^2 x_1^2 + 2g_{11}g_{12}x_1x_2 + g_{12}^2 x_2^2\,,\; g_{21}^2 x_1^2 + 2g_{21}g_{22}x_1x_2 + g_{22}^2 x_2^2, \\ & 2g_{21}g_{11}(g_{22}g_{11} - g_{21}g_{12})x_1x_2 \;+\; (g_{22}g_{11} - g_{21}g_{12})(g_{21}g_{12} + g_{22}g_{11})x_2^2, \\ & (g_{22}g_{11} - g_{21}g_{12})^3 x_2^3 \}. \end{aligned}$$

The group $GL_2(\Bbbk)$ decomposes into only two equivalence classes in this case:

- $\mathsf{in}_<(g \cdot I) = \langle x_1^2, x_2^2 \rangle$ if $g_{11}g_{21} = 0$
- $\mathsf{in}_<(g \cdot I) = \langle x_1^2, x_1x_2, x_2^3 \rangle$ if $g_{11}g_{21} \neq 0$

The second ideal is the generic initial ideal: $\mathsf{gin}(I) = \langle x_1^2, x_1x_2, x_2^3 \rangle$. ◇

The punch line is the result of Galligo, Bayer, and Stillman describing a general procedure to turn arbitrary ideals into Borel-fixed ideals.

Theorem 2.9 *The generic initial ideal* $\mathsf{gin}_<(I)$ *is Borel-fixed.*

Proof. We refer to Eisenbud's commutative algebra textbook, where this result appears as [Eis95, Theorem 15.20]. A complete proof is given there. □

It is important to note that the generic initial ideal $\mathsf{gin}_<(I)$ depends heavily on the choice of the term order $<$. Two extreme examples of term orders are the *purely lexicographic term order*, denoted $<_{\mathrm{lex}}$, and the *reverse lexicographic term order*, denoted $<_{\mathrm{revlex}}$. For two monomials $\mathbf{x}^{\mathbf{a}}$ and $\mathbf{x}^{\mathbf{b}}$ of the same degree, we have $\mathbf{x}^{\mathbf{a}} >_{\mathrm{lex}} \mathbf{x}^{\mathbf{b}}$ if the leftmost nonzero entry of the vector $\mathbf{a} - \mathbf{b}$ is positive, whereas $\mathbf{x}^{\mathbf{a}} >_{\mathrm{revlex}} \mathbf{x}^{\mathbf{b}}$ if the rightmost nonzero entry of the vector $\mathbf{a} - \mathbf{b}$ is negative.

Example 2.10 Let $f, g \in \Bbbk[x_1, x_2, x_3, x_4]$ be generic forms of degrees d and e, respectively. Considering the three smallest nontrivial cases, we list the generic initial ideal of $I = \langle f, g\rangle$ for both the lexicographic order and the reverse lexicographic order. The ideals $J = \mathsf{gin}_{\text{lex}}(I)$ are:

$$\begin{aligned}
(d,e) = (2,2) \quad J &= \langle x_2^4, x_1x_3^2, x_1x_2, x_1^2\rangle \\
&= \langle x_1, x_2^4\rangle \cap \langle x_1^2, x_2, x_3^2\rangle, \\
(d,e) = (2,3) \quad J &= \langle x_2^6, x_1x_3^6, x_1x_2x_4^4, x_1x_2x_3x_4^2, x_1x_2x_3^2, x_1x_2^2, x_1^2\rangle \\
&= \langle x_1, x_2^6\rangle \cap \langle x_1^2, x_2, x_3^6\rangle \cap \langle x_1^2, x_2^2, x_3, x_4^4\rangle \cap \langle x_1^2, x_2^2, x_3^2, x_4^2\rangle, \\
(d,e) = (3,3) \quad J &= \langle x_2^9, x_1x_3^{18}, x_1x_2x_4^{16}, x_1x_2x_3x_4^{14}, \ldots, x_1^3\rangle \text{ (26 generators).}
\end{aligned}$$

On the other hand, the ideals $J = \mathsf{gin}_{\text{revlex}}(I)$ are:

$$\begin{aligned}
(d,e) = (2,2) \qquad J &= \langle x_2^3, x_1x_2, x_1^2\rangle \\
&= \langle x_1, x_2^3\rangle \cap \langle x_1^2, x_2\rangle, \\
(d,e) = (2,3) \qquad J &= \langle x_2^4, x_1x_2^2, x_1^2\rangle \\
&= \langle x_1, x_2^4\rangle \cap \langle x_1^2, x_2^2\rangle, \\
(d,e) = (3,3) \qquad J &= \langle x_2^5, x_1x_2^3, x_1^2x_2, x_1^3\rangle \\
&= \langle x_1, x_2^5\rangle \cap \langle x_1^2, x_2^3\rangle \cap \langle x_1^3, x_2\rangle,
\end{aligned}$$

The reverse lex gin is much nicer than the lex gin, mostly because there are fewer generators, but also because they have lower degrees. All six ideals J above are Borel-fixed. $\diamond$

Let us conclude this section with one more generality on Gröbner bases: they work for submodules of free S-modules. Suppose that $\mathcal{F} = S^\beta$ is a free module of rank β, with basis $\mathbf{e}_1, \ldots, \mathbf{e}_\beta$. There is a general definition of term order for $\mathcal{F}$, which is a total order on elements of the form $m\mathbf{e}_i$, for monomials $m \in S$, satisfying appropriate analogues of the multiplicative and artinian properties of term orders for S. Initial modules are defined just as they were for ideals (which constitute the case $\beta = 1$). For our purposes, we need only consider term orders on $\mathcal{F}$ obtained from a term order on S by ordering the basis vectors $\mathbf{e}_1 > \cdots > \mathbf{e}_\beta$. To get such a term order, we have to pick which takes precedence, the term order on S or the ordering on the basis vectors. In the former case, we get the TOP order, which stands for *term-over-position*; in the latter case, we get the POT order, for *position-over-term*. In the POT order, for example, $m\mathbf{e}_i > m'\mathbf{e}_j$ if either $i < j$, or else $i = j$ and $m > m'$. If $M \subseteq \mathcal{F}$ is a submodule, then $\{f_1, \ldots, f_r\} \subset M$ is a *Gröbner basis* if $\mathsf{in}(f_1), \ldots, \mathsf{in}(f_r)$ generate $\mathsf{in}(M)$. The notion of reduced Gröbner basis for modules requires only that if $\mathsf{in}(f_k) = m\mathbf{e}_i$, then m does not divide m' for any other term $m'\mathbf{e}_i$ with the same $\mathbf{e}_i$ appearing in any f_j.

2.3 The Eliahou–Kervaire resolution

Next we describe the minimal free resolution, Betti numbers and Hilbert series of a Borel-fixed ideal I. The same construction works also for the larger class of so-called "stable ideals", but we restrict ourselves to the Borel-fixed case here. Throughout this section, the monomials $m_1, \ldots, m_r$ *minimally* generate the Borel-fixed ideal I, and for every monomial m, we write $\max(m)$ for the largest index of a variable dividing m. For instance, $\max(x_1^7 x_2^3 x_4^5) = 4$ and $\max(x_2 x_3^7) = 3$. Similarly, let $\min(m)$ denote the smallest index of a variable dividing m.

Lemma 2.11 *Each monomial m in the Borel-fixed ideal $I = \langle m_1, \ldots, m_r \rangle$ can be written uniquely as a product $m = m_i m'$ with $\max(m_i) \leq \min(m')$.*

In what follows, we abbreviate $u_i = \max(m_i)$ for $i = 1, \ldots, r$.

Proof. Uniqueness: Suppose $m = m_i m'_i = m_j m'_j$ both satisfy the condition, with $u_i \leq u_j$. Then m_i and m_j agree in every variable with index $< u_i$. If x_{u_i} divides m'_j, then $u_i = u_j$ by the assumed condition, whence one of m_i and m_j divides the other, so $i = j$. Otherwise, x_{u_i} does not divide m'_j. In this case the degree of x_{u_i} in m_i is at most the degree of x_{u_i} in m_j, which equals the degree of x_{u_i} in m, so that again m_i divides m_j and $i = j$.

Existence: Suppose that $m = m_j m'$ for some j, but that $u_j > u := \min(m')$. Proposition 2.3 says that we can replace m_j by any minimal generator m_i dividing $m_j x_u / x_{u_j}$. By construction, $u_i \leq u_j$, so either $u_i < u_j$, or $u_i = u_j$ and the degree of x_{u_i} in m_i is $<$ the degree of x_{u_i} in m_j. This shows that we cannot keep going on making such replacements forever. □

Recall that a quotient of S by a monomial ideal I has a *K-polynomial* if the $\mathbb{N}^n$-graded Hilbert series of S/I agrees with a rational function having denominator $(1-x_1)\cdots(1-x_n)$, in which case $\mathcal{K}(S/I;\mathbf{x})$ is the numerator.

Proposition 2.12 *For the Borel-fixed ideal $I = \langle m_1, \ldots, m_r \rangle$, the quotient S/I has K-polynomial*

$$\mathcal{K}(S/I;\mathbf{x}) \quad = \quad 1 - \sum_{i=1}^{r} m_i \prod_{j=1}^{u_i - 1} (1 - x_j).$$

Proof. By Lemma 2.11, the set of monomials in I is the disjoint union over $i = 1, \ldots, r$ of the monomials in $m_i \cdot \mathbb{k}[x_{u_i}, \ldots, x_n]$. The sum of all monomials in such a translated subalgebra of S equals the series

$$\frac{m_i}{\prod_{l=1}^{n}(1-x_l)} \prod_{j=1}^{u_i - 1} (1 - x_j)$$

by Example 1.11. Summing this expression from $i = 1$ to r yields the Hilbert series of I, and subtracting this from the Hilbert series of S yields the Hilbert series of S/I. Clear denominators to get the K-polynomial. □

Example 2.13 Let I be the ideal in Example 2.4. Its K-polynomial is

$$\begin{aligned}\mathcal{K}(S/I;\mathbf{x}) &= 1 - x_1^2 - x_1x_2(1-x_1) - x_2^3(1-x_1) - x_1x_3^3(1-x_1)(1-x_2)\\ &= 1 - x_1^2 - x_1x_2 - x_2^3 - x_1x_3^3\\ &\quad + x_1^2x_3^3 + x_1x_2x_3^3 + x_1x_2^3 + x_1^2x_2\\ &\quad - x_1^2x_2x_3^3.\end{aligned}$$

This expansion suggests that the minimal resolution of S/I has the form

$$0 \leftarrow S \longleftarrow S^4 \longleftarrow S^4 \longleftarrow S \leftarrow 0,$$

and this is indeed the case, by the formula in Theorem 2.18. ◇

The simplicial complexes that arise in connection with Borel-fixed ideals have rather simple geometry. Since we will need this geometry in the proof of Theorem 2.18, via Lemma 2.15, let us make a formal definition.

Definition 2.14 A simplicial complex Δ on the vertices $1,\ldots,k$ is **shifted** if $(\tau \smallsetminus \alpha) \cup \beta$ is a face of Δ whenever τ is a face of Δ and $1 \leq \alpha < \beta \leq k$.

The distinction between faces and facets will be crucial in what follows.

Lemma 2.15 *Fix a shifted simplicial complex* Γ *on* $1,\ldots,k$*, and let* $\Delta \subseteq \Gamma$ *consist of the faces of* Γ *not having* k *as a vertex. Then* $\dim_{\Bbbk} \widetilde{H}_i(\Gamma;\Bbbk)$ *equals the number of dimension* i *facets* τ *of* Δ *such that* $\tau \cup k$ *is not a face of* Γ.

Proof. Γ is a subcomplex of the cone $k * \Delta$ from the vertex k over Δ. By Definition 2.14, if $\tau \in \Delta$ is a face, then Γ contains every proper face of the simplex $\tau \cup k$. In other words, Γ is a *near-cone* over Δ, which is by definition obtained from $k * \Delta$ by removing the interior of the simplex $\tau \cup k$ for some of the facets τ of Δ.

The only i-faces of Γ are (i) the i-faces of Δ, (ii) the cones $\sigma \cup k$ over some subset of the $(i-1)$-facets $\sigma \in \Delta$, and (iii) the cones from k over all non-facet $(i-1)$-faces of Δ. If σ is an $(i-1)$-facet of Δ, then $\sigma \cup k \in \Gamma$ cannot have nonzero coefficient $c \in \Bbbk$ in any i-cycle of Γ, because σ would have coefficient $\pm c$ in its boundary.

For each $j \geq 0$, let $\Delta_j \subseteq \Delta$ be the subcomplex that is the union of all (closed) j-faces of Δ. For the purpose of computing $\widetilde{H}_i(\Gamma;\Bbbk)$, we assume using the previous paragraph that Δ has no facets of dimension less than i, by replacing Δ with $\Delta_{\geq i} = \bigcup_{j \geq i} \Delta_j$ and taking only those faces of Γ contained in $k * \Delta_{\geq i}$. Thus every i-face of $k * \Delta$ lies in Γ. Since we are interested in the i^{th} homology of Γ, we also assume that $\dim(\Delta) \leq i+1$.

There can be $(i+1)$-faces of Γ that do not lie in the cone $k*\Delta$, but these missing $(i+1)$-faces all have the form $\tau \cup k$ for a facet τ of dimension i in Δ. Now consider the long exact homology sequence arising from the inclusion $\Gamma \to k * \Delta$. It contains the sequence $\widetilde{H}_{i+1}(k*\Delta) \to \widetilde{H}_{i+1}(k*\Delta,\Gamma) \to \widetilde{H}_i(\Gamma) \to \widetilde{H}_i(k*\Delta)$. The outer terms are zero because $k * \Delta$ is a cone.

When Γ is the *minimal* near-cone over Δ, the dimension of the relative homology $\widetilde{H}_{i+1}(k * \Delta, \Gamma)$ is the number of i-facets of Δ, because the only faces of $k*\Delta$ contributing to the relative chain complex are $\tau \cup k$ for i-facets τ of Δ. Hence the isomorphism $\widetilde{H}_{i+1}(k * \Delta, \Gamma) \to \widetilde{H}_i(\Gamma)$ proves the lemma in this case. For general Γ, adding a face $\tau \cup k$ can only cancel at most one i^{th} homology class of Γ, so it must cancel exactly one, because adding all of the faces $\tau \cup k$ for i-facets of Δ yields $k * \Delta$, which has no homology. □

The main theorem of this section refers to an important notion that will resurface again in Chapter 5. For any vector $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$, let $|\mathbf{b}| = b_1 + \cdots + b_n$.

Definition 2.16 An $\mathbb{N}^n$-graded free resolution $\mathcal{F}_{\bullet}$ is **linear** if there is a choice of monomial matrices for the differentials of $\mathcal{F}_{\bullet}$ such that in each matrix, $|\mathbf{a}_p - \mathbf{a}_q| = 1$ whenever the scalar entry λ_{qp} is nonzero. A module M has **linear free resolution** if its minimal free resolution is linear.

Using the ungraded notation for maps between free S-modules, a $\mathbb{Z}$-graded free resolution is *linear* if the nonzero entries in some choice of matrices for all of its differentials are linear forms. When the resolution is $\mathbb{N}^n$-graded, the linear forms can be taken to be scalar multiples of variables.

Example 2.17 Let M be an $\mathbb{N}^n$-graded module whose generators all lie in degrees $\mathbf{b} \in \mathbb{N}^n$ satisfying $|\mathbf{b}| = d$ for some fixed integer $d \in \mathbb{N}$. Then M has linear resolution if and only if for all $i \geq 0$, the minimal i^{th} syzygies of M lie in degrees $\mathbf{b} \in \mathbb{N}^n$ satisfying $|\mathbf{b}| = d + i$. ◇

Theorem 2.18 *Let M be the module of first syzygies on the Borel-fixed ideal $I = \langle m_1, \ldots, m_r \rangle$. Then M has a Gröbner basis such that its initial module $\mathsf{in}(M)$ has linear free resolution. Moreover, $S^r/\mathsf{in}(M)$ has the same number of minimal i^{th} syzygies as $I \cong S^r/M$, namely $\sum_{j=1}^{r} \binom{\max(m_j)-1}{i}$.*

Proof. The idea of the proof is to compare the minimal free resolution of M to a direct sum of Koszul complexes. We make the following crucial labeling assumption, in which $\deg_u(m)$ is the degree of x_u in each monomial m, and again $u_i = \max(m_i)$ for $i = 1, \ldots, r$:

$$i > j \quad \Rightarrow \quad u_i \leq u_j \quad \text{and} \quad \deg_{u_j}(m_i) \leq \deg_{u_j}(m_j).$$

Let us begin by constructing some special elements in the syzygy module M. Consider any product $m = x_u m_j$ in which $u < u_j$. By Lemma 2.11, this monomial can be rewritten uniquely as

$$m \;=\; x_u \cdot m_j \;=\; m' \cdot m_i \quad \text{with} \quad u_i \leq \min(m').$$

Since $u < u_j$, we must have $\min(m') \leq u_j$. Moreover, if $\min(m') = u_j$, then $\deg_{u_j}(m_i) < \deg_{u_j}(m_j)$. Therefore $i < j$ with our labeling assumption. This means that the following vector is a nonzero first syzygy on I:

$$\underline{x_u \cdot \mathbf{e}_j} \;-\; m' \cdot \mathbf{e}_i \;\in\; M. \tag{2.1}$$

Fix any term order on S^r that picks the underlined term as the leading term for every $j = 1, \ldots, r$ and $u = 1, \ldots, u_j$; the POT order induced by $\mathbf{e}_1 > \mathbf{e}_2 > \cdots > \mathbf{e}_r$ will do, for instance. We claim that the set of syzygies (2.1), as u and j run over all pairs satisfying $u < u_j$, equals the reduced Gröbner basis of M, and in particular, generates M.

If the Gröbner basis property does not hold, then some nonzero syzygy

$$m'' \cdot \mathbf{e}_j \;-\; m' \cdot \mathbf{e}_i \;\in\; M$$

has the property that neither $m'' \cdot \mathbf{e}_j$ nor $m' \cdot \mathbf{e}_i$ lies in the submodule of S^r generated by the underlined leading terms in (2.1). This means that

$$\min(m') \geq \max(m_i) \quad \text{and} \quad \min(m'') \geq \max(m_j).$$

The identity $m' \cdot m_i = m'' \cdot m_j$ contradicts the uniqueness statement in Lemma 2.11. This contradiction proves that the relations (2.1) constitute a Gröbner basis for the submodule $M \subset S^r$. This Gröbner basis is reduced because no leading term $x_u \mathbf{e}_j$ divides either term of another syzygy (2.1).

We have shown that the initial module $\mathsf{in}(M)$ under the given term order is minimally generated by the monomials $x_u \cdot \mathbf{e}_j$ for which $u < u_j$. Hence this initial module decomposes as the direct sum

$$\mathsf{in}(M) \quad = \quad \bigoplus_{j=1}^{r} \langle x_1, x_2, \ldots, x_{u_j - 1} \rangle \cdot \mathbf{e}_j. \tag{2.2}$$

The minimal free resolution of $\mathsf{in}(M)$ is the direct sum of the minimal free resolutions of the r summands in (2.2). The minimal free resolution of the ideal $\langle x_1, x_2, \ldots, x_{u_j-1} \rangle$ is a Koszul complex, which is itself a linear resolution. Moreover, the number of i^{th} syzygies in this Koszul complex equals $\binom{u_j - 1}{i}$. We conclude that $\mathsf{in}(M)$ has linear resolution and that its number of minimal i^{th} syzygies equals the desired number, namely $\sum_{j=1}^{r} \binom{u_j - 1}{i}$.

We have reduced Theorem 2.18 to the claim that the Betti numbers of M equal those of its initial module $\mathsf{in}(M)$ in every degree $\mathbf{b} \in \mathbb{N}^n$. In fact, we only need to show that $\beta_{i,\mathbf{b}}(M) \geq \beta_{i,\mathbf{b}}(\mathsf{in}(M))$, because it is always the case that $\beta_{i,\mathbf{b}}(M) \leq \beta_{i,\mathbf{b}}(\mathsf{in}(M))$ for all $\mathbf{b} \in \mathbb{N}^n$ (we shall prove this in a general context in Theorem 8.29). Fix $\mathbf{b} = (b_1, \ldots, b_n)$ with $\beta_{i,\mathbf{b}}(\mathsf{in}(M)) \neq 0$, and let k be the largest index with $b_k > 0$.

By (2.2), the Betti number $\beta_{i,\mathbf{b}}(\mathsf{in}(M))$ equals the number of indices $j \in \{1, \ldots, r\}$ such that $\mathbf{x}^{\mathbf{b}}/m_j$ is a squarefree monomial $\mathbf{x}^\tau \in S$ for some subset $\tau \subseteq \{1, \ldots, u_j - 1\}$ of size $i + 1$. All of these indices j share the property that $\deg_{x_k}(m_j) = b_k$. Each index j arising here leads to a different $(i+1)$-subset τ of $\{1, \ldots, k-1\}$.

The Betti number $\beta_{i,\mathbf{b}}(M) = \beta_{i+1,\mathbf{b}}(I)$ can be computed, by Theorem 1.34, as the dimension of the i^{th} homology group of the upper Koszul simplicial complex $K^{\mathbf{b}}(I)$ in degree $\mathbf{b}$. Applying Proposition 2.3 to monomials $m = \mathbf{x}^{\mathbf{b}-\tau}$ for squarefree vectors τ, we find that $K^{\mathbf{b}}(I)$ is shifted. Hence we deduce from Lemma 2.15 that $\dim_{\Bbbk} \widetilde{H}_i(K^{\mathbf{b}}(I); \Bbbk)$ equals the number of dimension i facets $\tau \in \Delta$ such that $\tau \cup k$ is not a face of $K^{\mathbf{b}}(I)$. But every size $i+1$ subset τ from the previous paragraph is a facet of Δ, and $\tau \cup k$ is not in $K^{\mathbf{b}}(I)$, both because $\mathbf{x}^{\mathbf{b}-\tau} = m_i$ is a minimal generator of I. Therefore $\beta_{i,\mathbf{b}}(M) \geq \beta_{i,\mathbf{b}}(\mathsf{in}(M))$, and the proof is complete. $\square$

We illustrate Theorem 2.18 and its proof with two nontrivial examples.

Example 2.19 Let $n = 4$ and $r = 7$, and consider the following ideal:

$$
\begin{array}{ccccccc}
\langle x_1x_2x_4^4, & x_1x_2x_3x_4^2, & x_1x_3^6, & x_1x_2x_3^2, & x_2^6, & x_1x_2^2, & x_1^2\rangle. \\
x_3\,\mathbf{e}_1 & -x_4^2\,\mathbf{e}_2 & & & & & \\
x_2\,\mathbf{e}_1 & & & & & -x_4^4\,\mathbf{e}_6 & \\
x_1\,\mathbf{e}_1 & & & & & & -x_2x_4^4\,\mathbf{e}_7 \\
 & x_3\,\mathbf{e}_2 & & -x_4^2\,\mathbf{e}_4 & & & \\
 & x_2\,\mathbf{e}_2 & & & & -x_3x_4^2\,\mathbf{e}_6 & \\
 & x_1\,\mathbf{e}_2 & & & & & -x_2x_3x_4^2\,\mathbf{e}_7 \\
 & & x_2\,\mathbf{e}_3 & -x_3^4\,\mathbf{e}_4 & & & \\
 & & x_1\,\mathbf{e}_3 & & & & -x_3^6\,\mathbf{e}_7 \\
 & & & x_2\,\mathbf{e}_4 & & -x_3^2\,\mathbf{e}_6 & \\
 & & & x_1\,\mathbf{e}_4 & & & -x_2x_3^2\,\mathbf{e}_7 \\
 & & & & x_1\,\mathbf{e}_5 & -x_2^4\,\mathbf{e}_6 & \\
 & & & & & x_1\,\mathbf{e}_6 & -x_2^2\,\mathbf{e}_7
\end{array}
$$

This monomial ideal is Borel-fixed. Beneath the seven generators, we wrote in 12 rows the 12 minimal first syzygies (2.1) on the generators. These form a Gröbner basis for the syzygy module M, and the initial module is

$$
\begin{array}{rcl}
\mathsf{in}(M) & = & \langle x_1\,\mathbf{e}_1, \quad x_2\,\mathbf{e}_1, \quad x_3\,\mathbf{e}_1, \\
 & & x_1\,\mathbf{e}_2, \quad x_2\,\mathbf{e}_2, \quad x_3\,\mathbf{e}_2, \\
 & & x_1\,\mathbf{e}_3, \quad x_2\,\mathbf{e}_3, \\
 & & x_1\,\mathbf{e}_4, \quad x_2\,\mathbf{e}_4, \\
 & & x_1\,\mathbf{e}_5, \\
 & & x_1\,\mathbf{e}_6\rangle \\
 & \subset & S^7 = \Bbbk[x_1, x_2, x_3, x_4]^7.
\end{array}
$$

Its minimal free resolution is a direct sum of six Koszul complexes:

$$
\begin{array}{rcccccccc}
(S\,\mathbf{e}_1 & \longleftarrow & S^3 & \longleftarrow & S^3 & \longleftarrow & S & \longleftarrow & 0) \\
\oplus\ (S\,\mathbf{e}_2 & \longleftarrow & S^3 & \longleftarrow & S^3 & \longleftarrow & S & \longleftarrow & 0) \\
\oplus\ (S\,\mathbf{e}_3 & \longleftarrow & S^2 & \longleftarrow & S & \longleftarrow & 0) & & \\
\oplus\ (S\,\mathbf{e}_4 & \longleftarrow & S^2 & \longleftarrow & S & \longleftarrow & 0) & & \\
\oplus\ (S\,\mathbf{e}_5 & \longleftarrow & S & \longleftarrow & 0) & & & & \\
\oplus\ (S\,\mathbf{e}_6 & \longleftarrow & S & \longleftarrow & 0) & & & &
\end{array}
$$

$$
0 \longleftarrow \mathsf{in}(M) \longleftarrow S^{12} \longleftarrow S^8 \longleftarrow S^2 \longleftarrow 0.
$$

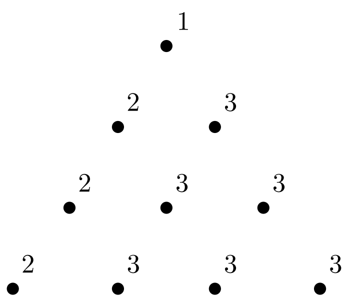
The resolution of $\mathsf{in}(M)$ is linear and lifts (by adding trailing terms as in Schreyer's algorithm [Eis95, Theorem 15.10]) to the minimal free resolution of M. The resulting resolution of the Borel-fixed ideal S^7/M is called the *Eliahou–Kervaire resolution*:

$$0 \leftarrow S \xleftarrow{(x_1x_2x_4^4 \quad x_1x_2x_3x_4^2 \quad \cdots \quad x_1^2)} S^7 \longleftarrow S^{12} \longleftarrow S^8 \longleftarrow S^2 \leftarrow 0.$$

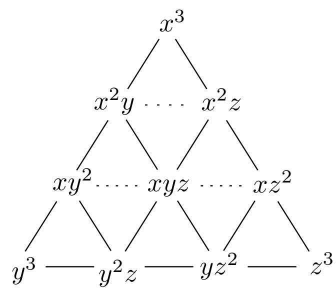
The reader is encouraged to compute the matrices representing the differentials in a computer algebra system. ◇

Our results on the Betti numbers of Borel-fixed ideals apply in particular to the $GL_n(\Bbbk)$-fixed ideals. By Corollary 2.2, these are the powers $\mathfrak{m}^d$ of the maximal homogeneous ideal $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$, as follows when $n = d = 3$.

Example 2.20 Let $n = d = 3$, and use the variable set $\{x, y, z\}$. The Betti numbers and Eliahou–Kervaire resolution of the Borel-fixed ideal $I = \langle x, y, z \rangle^3$ can be visualized as follows:



$\max(m_i)$



$\langle x, y, z \rangle^3$

The importance of the dotted lines in the right-hand diagram will be explained in Example 4.22. The numbers in the left-hand diagram determine the binomial coefficients $\binom{\max(m_j)-1}{i}$ from Theorem 2.18, which are given in the triangles below. By adding these triangles, we get the Betti numbers of the minimal free resolution

$$\begin{array}{ccccccccc} S & \longleftarrow & S^{10} & \longleftarrow & S^{15} & \longleftarrow & S^{6} & \longleftarrow & 0 \\ & & \begin{smallmatrix} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 1 & & 1 & \\ 1 & & 1 & & 1 & & 1 \end{smallmatrix} & & \begin{smallmatrix} & & & 0 & & & \\ & & 1 & & 2 & & \\ & 1 & & 2 & & 2 & \\ 1 & & 2 & & 2 & & 2 \end{smallmatrix} & & \begin{smallmatrix} & & & 0 & & & \\ & & 0 & & 1 & & \\ & 0 & & 1 & & 1 & \\ 0 & & 1 & & 1 & & 1 \end{smallmatrix} & & \end{array}$$

The triangles show how the resolution of the initial module $\mathsf{in}(M)$ decomposes as a direct sum of 10 Koszul complexes, one for each generator of I. ◇

2.4 Lex-segment ideals

In this section, fix the lexicographic term order $< \; = \; <_{\text{lex}}$ on the polynomial ring $S = \Bbbk[x_1, \ldots, x_n]$. The d^{th} graded component S_d will be identified with the set of all monomials in S of degree d. Fix a function $H : \mathbb{N} \to \mathbb{N}$ that equals the $\mathbb{N}$-graded *Hilbert function* of some homogeneous ideal I in S, meaning that $H(d)$ is the number of $\Bbbk$-linearly independent homogeneous polynomials of degree d lying in the ideal I. There are many choices for I, given our fixed H, and this section is about a certain extreme choice.

Let L_d be the vector space over $\Bbbk$ spanned by the $H(d)$ largest monomials in the lexicographic order on S_d. Define a subspace of S by taking the direct sum of these finite-dimensional spaces of homogeneous polynomials:

$$L \quad = \quad \bigoplus_{d=0}^{\infty} L_d.$$

The following result is due to Macaulay [Mac27].

Proposition 2.21 *The graded vector space L is an ideal, called the **lex-segment ideal** for the Hilbert function H.*

A proof of this proposition will be given later, as part of our general combinatorial development in this section. It follows from Proposition 2.3 that L is Borel-fixed. The reason for studying lex-segment ideals is because their numerical behavior is so extreme that they bound from above the numerical behavior of all other ideals. The seminal result along these lines is the following classical theorem of Macaulay.

Theorem 2.22 (Macaulay's Theorem) *For every degree $d \geq 0$, the lex-segment ideal L for the Hilbert function H has at least as many generators in degree d as every other (monomial) ideal with Hilbert function H.*

Example 2.23 Let $n = 4$ and let H be the Hilbert function of the ideal generated by two generic forms of degrees d and e. The lex-segment ideal L for this Hilbert function has more generators than the lexicographic initial ideal in Example 2.10. The first two ideals in this family are

$$\begin{aligned}
(d,e) = (2,2): \quad L \quad &= \quad \langle x_2^4x_3^2, x_2^5, x_1x_4^4, x_1x_3x_4^2, x_1x_3^2, x_1^2, x_1x_2\rangle,\\
(d,e) = (2,3): \quad L \quad &= \quad \langle x_2^6x_3^6, x_2^7x_4^4, x_2^7x_3x_4^2, x_2^9, x_2^8x_3, x_2^7x_3^2, x_2^8x_4, x_1x_3^2x_4^5,\\
&\qquad x_1x_3x_4^6, x_1x_4^7, x_1x_3^4x_4^2, x_1x_3^3x_4^3, x_1x_3^5, x_1x_2x_4^4,\\
&\qquad x_1x_2x_3x_4^2, x_1x_2x_3^2, x_1x_2^2, x_1^2\rangle.
\end{aligned}$$

How many monomial generators does L have for $(d,e) = (3,3)$? $\diamond$

In Theorem 2.22, it is enough to restrict our attention to monomial ideals, since any initial ideal of an $\mathbb{N}$-graded ideal I has a least as many generators in each degree d as I does. In fact, in view of Theorem 2.9 on generic initial ideals, it suffices to consider only Borel-fixed monomial ideals, as $g \cdot I$ has the same number of generators in each degree as I does.

The degrees of the generators of an ideal measure its zeroth Betti numbers. One can also ask which ideals have the worst behavior with respect to the degrees of the higher Betti numbers. The ultimate statement is that lex-segment ideals take the cake simultaneously for all Betti numbers.

Theorem 2.24 (Bigatti–Hulett Theorem) *For every $i \in \{0, 1, \ldots, n\}$ and $d \geq 0$, the lex-segment ideal L has the most degree d minimal i^{th} syzygies among all (monomial) ideals I with the same fixed Hilbert function H.*

In this section we present proofs for Theorems 2.22 and 2.24 and, of course, also for Proposition 2.21. For the Bigatti–Hulett Theorem, it also suffices to consider only Borel-fixed monomial ideals I. The reason is that Betti numbers can only increase when we pass to an initial ideal (we will prove this in Theorem 8.29), and generic initial ideals are Borel-fixed. To begin with, we need to introduce some combinatorial definitions.

Let W be any finite set of monomials in the polynomial ring S, and write $W_d = W \cap S_d$ for the subset of monomials in W of degree d. For $i \in \{1, \ldots, n\}$, set

$$\begin{aligned} \mu_i(W) &= \big|\{m \in W \mid \max(m) = i\}\big|, \\ \mu_{\leq i}(W) &= \big|\{m \in W \mid \max(m) \leq i\}\big|. \end{aligned}$$

Call W a *Borel set* of monomials if $mx_i/x_j \in W$ whenever x_j divides $m \in W$ and $i < j$. We call W a *lex segment* if $m \in W$ and $m' >_{\text{lex}} m$ implies $m' \in W$. If W is a Borel set then, by Lemma 2.11, every monomial m in $\{x_1, \ldots, x_n\} \cdot W$ factors uniquely as $m = x_i \cdot \tilde{m}$ for some $\tilde{m} \in W$ with $\max(\tilde{m}) \leq i$. This implies the following identity, which holds for all Borel sets W and all $i \in \{1, \ldots, n\}$:

$$\mu_i(\{x_1, \ldots, x_n\} \cdot W) \quad = \quad \mu_{\leq i}(W). \tag{2.3}$$

In the next lemma, we consider sets of monomials all having equal degree d.

Lemma 2.25 *Let L be a lex segment in S_d and B a Borel set in S_d. If $|L| \leq |B|$ then $\mu_{\leq i}(L) \leq \mu_{\leq i}(B)$ for all i.*

Proof. The prove is by induction on n. We distinguish three cases according to the value of i. If $i = n$ then the asserted inequality is obvious:

$$\mu_{\leq n}(L) \; = \; |L| \; \leq \; |B| \; = \; \mu_{\leq n}(B).$$

Suppose now that $i = n - 1$. Partition the Borel set B by powers of x_n:

$$B \quad = \quad B[0] \cup \big(x_n \cdot B[1]\big) \cup \big(x_n^2 \cdot B[2]\big) \cup \cdots \cup \big(x_n^d \cdot B[d]\big).$$

Then $B[i]$ is a Borel set in $\Bbbk[x_1, \ldots, x_{n-1}]_{d-i}$. Similarly, decompose the lex segment L, so $L[i]$ is a lex segment in $\Bbbk[x_1, \ldots, x_{n-1}]_{d-i}$. Let $C[i]$ denote the lex segment in $\Bbbk[x_1, \ldots, x_{n-1}]_{d-i}$ of the same cardinality as $B[i]$. Set

$$C \quad = \quad C[0] \cup \big(x_n \cdot C[1]\big) \cup \big(x_n^2 \cdot C[2]\big) \cup \cdots \cup \big(x_n^d \cdot C[d]\big).$$

By induction, Lemma 2.25 is true in $n-1$ variables, so we have inequalities

$$\mu_{\le j}(C[i]) \quad \le \quad \mu_{\le j}(B[i]) \quad \text{for all } i, j. \tag{2.4}$$

We claim that C is a Borel set. Since B is a Borel set, $\{x_1, \ldots, x_{n-1}\}B[i]$ is a subset of $B[i-1]$. The inductive hypothesis (2.4) together with (2.3) implies

$$\begin{aligned}
|\{x_1, \ldots, x_{n-1}\} \cdot C[i]| \quad &= \quad \sum_{j=1}^{n-1} \mu_j(\{x_1, \ldots, x_{n-1}\} \cdot C[i]) \quad = \quad \sum_{j=1}^{n-1} \mu_{\le j}(C[i]) \\
&\le \quad \sum_{j=1}^{n-1} \mu_{\le j}(B[i]) \\
&= \quad \sum_{j=1}^{n-1} \mu_j(\{x_1, \ldots, x_{n-1}\} \cdot B[i]) \\
&= \quad |\{x_1, \ldots, x_{n-1}\} \cdot B[i])| \\
&\le \quad |B[i-1]| \quad = \quad |C[i-1]|.
\end{aligned}$$

Since $\{x_1, \ldots, x_{n-1}\} \cdot C[i]$ and $C[i-1]$ are lex segments, we deduce that

$$\{x_1, \ldots, x_{n-1}\} \cdot C[i] \quad \subseteq \quad C[i-1],$$

which means that C is a Borel set in S_d.

Since L is a lex segment and since $|L| \le |B| = |C|$, the lexicographically minimal monomials in C and L respectively satisfy

$$\min_{\text{lex}}(C) \quad \le_{\text{lex}} \quad \min_{\text{lex}}(L).$$

Since both C and L are Borel-fixed, this implies that

$$\min_{\text{lex}}(C[0]) \quad \le_{\text{lex}} \quad \min_{\text{lex}}(L[0]).$$

Thus $L[0] \subseteq C[0]$ since both are lex segments in $\Bbbk[x_1, \ldots, x_{n-1}]_d$. Hence

$$\mu_{\le n-1}(L) \;=\; |L[0]| \;\le\; |C[0]| \;=\; |B[0]| \;=\; \mu_{\le n-1}(B), \tag{2.5}$$

which completes the proof for $i = n-1$.

Finally, consider the case $i \le n-2$. From (2.5) we have $|L[0]| \le |B[0]|$, so Lemma 2.25 can be applied inductively to the sets $B[0]$ and $L[0]$ to get

$$\mu_{\le i}(L) \;=\; \mu_{\le i}(L[0]) \;\le\; \mu_{\le i}(B[0]) \;=\; \mu_{\le i}(B) \quad \text{for } 1 \le i \le n-2.$$

Here, the middle inequality is the one from the inductive hypothesis. □

For any finite set W of monomials, define

$$\beta_i(W) \quad = \quad \sum_{m \in W} \binom{\max(m) - 1}{i}. \tag{2.6}$$

If W minimally generates a Borel-fixed ideal I, then according to Theorem 2.18, $\beta_i(W)$ is the number of minimal i^{th} syzygies of I. But certainly we can consider the combinatorial number $\beta_i(W)$ for any set of monomials.

Lemma 2.26 *If B is a Borel set in S_d then*

$$\beta_i(B) = \binom{n-1}{i} \cdot |B| - \sum_{j=1}^{n-1} \mu_{\leq j}(B) \binom{j-1}{i-1}.$$

Proof. Rewrite (2.6) for $W = B$ as follows:

$$\begin{aligned}
\beta_i(B) &= \sum_{j=1}^{n} \mu_j(B) \binom{j-1}{i} \\
&= \sum_{j=1}^{n} \big(\mu_{\leq j}(B) - \mu_{\leq j-1}(B)\big) \binom{j-1}{i} \\
&= \mu_{\leq n}(B) \binom{n-1}{i} + \sum_{j=1}^{n-1} \mu_{\leq j}(B) \binom{j-1}{i} - \sum_{j=2}^{n} \mu_{\leq j-1}(B) \binom{j-1}{i} \\
&= |B| \binom{n-1}{i} + \sum_{j=1}^{n-1} \mu_{\leq j}(B) \left(\binom{j-1}{i} - \binom{j}{i} \right).
\end{aligned}$$

The binomial identity $\binom{j-1}{i} - \binom{j}{i} = -\binom{j-1}{i-1}$ completes the proof. □

Lemma 2.27 *Let L be a lex segment in S_d and B a Borel set in S_d with $|L| = |B|$. Then the following inequalities hold:*

1. $\beta_i(L) \geq \beta_i(B)$.
2. $\beta_i(\{x_1, \ldots, x_n\} \cdot L) \leq \beta_i(\{x_1, \ldots, x_n\} \cdot B)$.

Proof. The proof of part 1 is immediate from Lemmas 2.25 and 2.26:

$$\begin{aligned}
\beta_i(L) &= \binom{n-1}{i} \cdot |L| - \sum_{j=1}^{n-1} \mu_{\leq j}(L) \binom{j-1}{i-1} \\
&\geq \binom{n-1}{i} \cdot |B| - \sum_{j=1}^{n-1} \mu_{\leq j}(B) \binom{j-1}{i-1} \\
&= \beta_i(B).
\end{aligned}$$

For part 2, apply the identity (2.3) for both B and L to get

$$\begin{aligned}
\beta_i(\{x_1, \ldots, x_n\} \cdot L) &= \sum_{j=1}^{n} \mu_j\big(\{x_1, \ldots, x_n\} \cdot L\big) \cdot \binom{j-1}{i} \\
&= \sum_{j=1}^{n} \mu_{\leq j}(L) \binom{j-1}{i} \\
&\leq \sum_{j=1}^{n} \mu_{\leq j}(B) \binom{j-1}{i} \\
&= \sum_{j=1}^{n} \mu_j\big(\{x_1, \ldots, x_n\} \cdot B\big) \cdot \binom{j-1}{i}.
\end{aligned}$$

This quantity equals $\beta_i(\{x_1, \ldots, x_n\} \cdot B)$, and the proof is complete. $\square$

We are now ready to tie up all loose ends and prove the three assertions.

Proof of Proposition 2.21. The function H is the Hilbert function of some ideal B, which we may assume to be Borel-fixed by Theorem 2.9, because Hilbert series are preserved under the operations $I \rightsquigarrow g \cdot I$ and $I \rightsquigarrow \mathsf{in}(I)$ (the latter uses that the standard monomials constitute a vector space basis modulo each of I and $\mathsf{in}(I)$). For any degree d, we have $|L_d| = |B_d|$. Using Lemma 2.25 and (2.3), we find that

$$\begin{aligned} |\{x_1, \ldots, x_n\} \cdot L_d| &= \sum_{j=1}^{n} \mu_j(\{x_1, \ldots, x_n\} \cdot L_d) \\ &= \sum_{j=1}^{n} \mu_{\leq j}(L_d) \\ &\leq \sum_{j=1}^{n} \mu_{\leq j}(B_d) \\ &= |\{x_1, \ldots, x_n\} \cdot B_d| \\ &\leq |B_{d+1}| \\ &= |L_{d+1}|. \end{aligned}$$

Both $\{x_1, \ldots, x_n\} \cdot L_d$ and L_{d+1} are lex segments in S_{d+1}. The inequality between their cardinalities implies the inclusion

$$\{x_1, \ldots, x_n\} \cdot L_d \quad \subseteq \quad L_{d+1}.$$

Since this holds for all d, we conclude that L is an ideal. $\square$

Proof of Theorem 2.22. For any graded ideal I, any term order, and any $d \geq 0$, the number of minimal generators of $\mathsf{in}(I)$ in degree d cannot be smaller than the number of minimal generators of I in degree d, because every Gröbner basis for I contains a minimal generating set. Therefore, replacing I with $\mathsf{gin}(I)$, we need only compare L to Borel-fixed ideals B.

In the previous proof, we derived the inequalities

$$|\{x_1, \ldots, x_n\} \cdot L_d| \;\leq\; |\{x_1, \ldots, x_n\} \cdot B_d| \;\leq\; |B_{d+1}| \;=\; |L_{d+1}|.$$

The number of minimal generators of L in degree $d+1$ is the difference $|L_{d+1}| - |\{x_1, \ldots, x_n\} \cdot L_d|$ between the outer two terms. The corresponding number for B is the difference $|B_{d+1}| - |\{x_1, \ldots, x_n\} \cdot B_d|$ between the middle two terms, which can only be smaller. This proves Macaulay's Theorem. $\square$

Next we rewrite the Eliahou–Kervaire formula for the Betti numbers of a Borel-fixed ideal I. If gens(I) is the set of minimal generators of I, then

$$\beta_i(\mathrm{gens}(I)) \quad = \quad \sum_{d>0} \big(\beta_i(I_d) - \beta_i(\{x_1, \ldots, x_n\} \cdot I_{d-1})\big). \tag{2.7}$$

Since I is finitely generated, all but finitely many terms in this sum cancel. Thus the right side of (2.7) reduces to the finite sum (2.6) for $W = \text{gens}(I)$.

Proof of Theorem 2.24. Let B be a Borel-fixed ideal and L the lex-segment ideal with the same Hilbert function as B. Our claim is the inequality

$$\beta_i(\text{gens}(B)) \leq \beta_i(\text{gens}(L)) \quad \text{for } i = 0, 1, \ldots, n.$$

Expanding both sides using (2.7), we find that the desired inequality follows immediately from parts 1 and 2 of Lemma 2.27. □

Exercises

2.1 Give necessary and sufficient conditions, in terms of $i_1, \ldots, i_r$ and $a_1, \ldots, a_r$, for an irreducible monomial ideal $I = \langle x_{i_1}^{a_1}, \ldots, x_{i_r}^{a_r} \rangle$ to be Borel-fixed.

2.2 Can you find a general formula for the number $\mathcal{B}(r, d)$ of Borel-fixed ideals generated by r monomials of degree d in three unknowns $\{x_1, x_2, x_3\}$?

2.3 Show that all associated primes of a Borel-fixed ideal are also Borel-fixed.

2.4 Is the class of Borel-fixed ideals closed under the ideal-theoretic operations of taking intersections, sums, and products?

2.5 Fix a term order on $\mathbb{k}[x_1, \ldots, x_n]$. Use the artinian property of term orders to show that every ideal has a unique reduced Gröbner basis. Do the same for submodules of free S-modules under any TOP or POT order.

2.6 Find a Borel-fixed ideal that is not the initial monomial ideal of any homogeneous prime ideal in $\mathbb{k}[x_1, \ldots, x_n]$. Are such examples rare or abundant?

2.7 Prove that if I is Borel fixed and $<$ is any term order, then $\mathsf{gin}_<(I) = I$.

2.8 Let $I = \langle x_1x_2, x_1x_3 \rangle$ and fix the lexicographic term order on $S = \mathbb{k}[x_1, x_2, x_3]$. List all distinct monomial ideals $\mathsf{in}_<(g \cdot I)$ as g runs over $GL_3(\mathbb{k})$. Find a comprehensive Gröbner basis $\mathcal{C}$ as in the proof of Lemma 2.6.

2.9 Let P be the *parabolic subgroup* of $GL_4(\mathbb{k})$ corresponding to the partition $4 = 2 + 2$, so P consists of all matrices of the form

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}.$$

Derive a combinatorial condition characterizing P-fixed ideals in $\mathbb{k}[x_1, x_2, x_3, x_4]$.

2.10 Let I be the ideal generated by two general homogeneous polynomials of degree 3 and 4 in $\mathbb{k}[x_1, x_2, x_3, x_4]$. Compute the generic initial ideal $\mathsf{gin}_<(I)$ for the lexicographic term order and for the reverse lexicographic term order. Also compute the lex-segment ideal with the same Hilbert function.

2.11 Let $I = \langle x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4 \rangle$. Compute the generic initial ideal $\mathsf{gin}_<(I)$ for the lexicographic and reverse lexicographic term orders. Also compute the lex-segment ideal with the same Hilbert function.

2.12 Compute the Betti numbers and Hilbert series of the ideal

$$I \quad = \quad \langle x_1, x_2, x_3, x_4, x_5 \rangle^5.$$

2.13 If $\mathcal{F}_{\bullet}$ is a linear free resolution, must *every* choice of matrices for its differentials have only linear forms for nonzero entries? Must $\mathcal{F}_{\bullet}$ be minimal?

2.14 Given a Borel-fixed ideal I, compute $K^{\mathbf{b}}(I)$ in any degree $\mathbf{b} \in \mathbb{N}^n$.

2.15 Let M be the first syzygy module of any Borel-fixed ideal. Give an example to show that even though $\mathsf{in}(M)$ has linear resolution, M itself need not. More generally, write down explicitly all of the boundary maps in the Eliahou–Kervaire resolution. Hint: Feel free to consult [EK90].

2.16 Is lexicographic order the only one for which Proposition 2.21 holds?

2.17 Can you find a monomial ideal that is not lex-segment but has the same graded Betti numbers as the lex-segment ideal with the same Hilbert function?

Notes

The original motivation for generic initial ideals, and hence Borel-fixed ideals, came from Hartshorne's proof of the connectedness of the Hilbert scheme of subschemes of projective space [Har66a]. Galligo proved Theorem 2.9 in characteristic zero [Gal74], and then Bayer and Stillman worked out the case of arbitrary characteristic [BS87]. It is worth noting that some of the other results in this chapter do not hold verbatim in positive characteristic, partially because the notion of "Borel-fixed" has a different combinatorial characterization due to Pardue [Par94]. See Eisenbud's textbook [Eis95, Section 15.9] for an exposition of Borel-fixed and generic initial ideals, including the finite characteristic case as well as more history and references.

The Eliahou–Kervaire resolution first appeared in [EK90], where it was derived for the class of *stable ideals*, which is slightly more general than Borel-fixed ideals. The passage from a monomial ideal to its generic initial ideals with respect to various term orders is called *algebraic shifting* in the combinatorics literature. This is an active area of research at the interface of combinatorics and commutative algebra; see the articles by Aramova–Herzog–Hibi [AHH00] and Babson–Novik–Thomas [BNT02] as well as the references given there. The explicit identification of cycles representing homology classes in shifted complexes, such as the boundaries of the missing faces $\tau \cup k$ in Lemma 2.15, is typical; in fact, it is a motivating aspect of their combinatorics (see [BK88, BK89], for example).

Theorem 2.22 is one of Macaulay's fundamental contributions to the theory of Hilbert functions [Mac27]. Theorem 2.24 is due independently to Bigatti [Big93] and Hulett [Hul93]; the proof given here is Bigatti's. The geometry of lexicographic generic initial ideals is a promising direction of future research, toward which first steps have been taken in recent work of Conca and Sidman [CS04].